On the homomorphisms between scalar generalized Verma modules

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Abstract

In this article, we study the homomorphisms between scalar generalized Verma modules. We conjecture that any homomorphism between is composition of elementary homomorphisms. The purpose of this article is to show the conjecture is affirmative for many parabolic subalgebras under the assumption that the infinitesimal characters are regular. [1]

§ 0. Introduction

We study the homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [4]).

Classification of the homomorphisms between scalar generalized Verma modules is equivalent to that of equivariant differential operators between the spaces of sections of homogeneous line bundles on generalized flag manifolds. (cf. [22],[12], [17], [9], and [16].)

In [35], Verma constructed homomorphisms between Verma modules associated with root reflections. Bernstein, I. M. Gelfand, and S. I. Gelfand proved that all the nontrivial homomorphisms between Verma modules are compositions of homomorphisms constructed by Verma. ([2])

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Later, Lepowsky studied the generalized Verma modules. In particular, Lepowsky ([**?**]) constructed a class of nontrivial homomorphisms between scalar generalized Verma modules associated to the parabolic subalgebras which are the complexifications of the minimal parabolic subalgebras of real reductive Lie algebras.

In [30], elementary homomorphisms between scalar generalized Verma modules are introduced. They can be regarded as a generalization of homomorphisms introduced by Verma and Lepowsky.

We propose a conjecture on the classification of the homomorphisms between scalar generalized Verma modules, which can be regarded as a generalization of the above-mentioned result of Bernstein-Gelfand-Gelfand.

Conjecture A All the nontrivial homomorphisms between scalar generalized Verma modules are compositions of elementary homomorphisms.

Soergel's result ([33] Theorem 11) implies that Conjecture A is reduced to the integral infinitesimal character case.

The purpose of this article is to show the conjecture is affirmative for many parabolic subalgebras under the assumption that the infinitesimal characters are regular. In order to explain our results, we introduce some notations. Let $\mathfrak{g}$ be a complex reductive Lie algebra and we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We denote by Δ (resp. W) the root system (resp. the Weyl group) with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix a basis Π of Δ. For $\Theta \subsetneq \Pi$, we put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall\alpha \in \Theta \ \alpha(H) = 0\}$ and $\Sigma_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Delta\} - \{0\}$. We denote by $\mathfrak{p}_\Theta$ the standard parabolic subalgebra corresponding to Θ and by $\mathfrak{l}_\Theta$ its Levi subalgebra containing $\mathfrak{h}$. We consider the Weyl group for parabolic sub algebra $W(\Theta) = \{w \in\mid w\Theta = \Theta\}$.

We call Θ normal if any two parabolic subalgebras with the Levi part $\mathfrak{l}_\Theta$ are conjugate under an inner automorphism of $\mathfrak{g}$. If Θ is normal, we call $\mathfrak{p}_\Theta$ normal. For example, a complexified minimal parabolic subalgebras of real simple Lie algebras except $\mathfrak{su}(p,q) \quad (p-1 > q > 0)$, $\mathfrak{so}^*(4n+2)$, $\mathfrak{e}_{6(-14)}$ are normal. Roughly speaking, if Θ is normal, the reflection σ_γ on $\mathfrak{a}_\Theta$ with respect to $\gamma \in \Sigma_\Theta$ can be regarded as an involution of the Weyl group for $(\mathfrak{g}, \mathfrak{h})$. A normal subset Θ of Π is called strictly normal, if σ_γ is a Duflo involution of some Weyl group (see Definition 4.2.1) . If Θ is strictly normal, there exists an elementary homomorphism with repect to σ_γ for each $\gamma \in \Sigma_\Theta$.

Let $\mathfrak{p}_\Theta$ be a complexified minimal parabolic subalgebra of a real simple Lie algebra and assume $\mathfrak{p}_\Theta$ is normal but is not strictly normal. Then, $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of $\mathfrak{so}(2n+1-q, q) \quad (n > q \geqslant 1)$, or $\mathfrak{sp}(n,n) \quad (n \geqslant 1)$.

One of the main result of this article is the following theorem.

Theorem B (Theorem 5.1.3) *If Θ is strictly normal, then each nontriv-*

ial homomorphism between scalar generalized Verma modules induced from $\mathfrak{p}_\Theta$ with regular integral infinitesimal character is composition of elementary homomorphisms.

We also have the following result.

Theorem C (cf. Theorem 5.1.3, Corollary 6.8.7, Proposition 6.5.2) *If Θ is normal and $\mathfrak{g}$ is an exceptional Lie algebra, then each nontrivial homomorphism between scalar generalized Verma modules induced from $\mathfrak{p}_\Theta$ with a regular integral infinitesimal character is composition of elementary homomorphisms.*

For $\mathfrak{gl}(n, \mathbb{C})$, we have the following result. (In fact, we have a more general result.)

Theorem D (Corollary 7.3.6) *If $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of $\mathfrak{su}(p, q)$, then each nontrivial homomorphism between scalar generalized Verma modules induced from $\mathfrak{p}_\Theta$ with a regular integral infinitesimal character is composition of elementary homomorphisms.*

Finally, we have the following result.

Corollary E (Theorem 5.1.3, Corollary 6.3.7) *Let $\mathfrak{p}_\Theta$ be a complexified minimal parabolic subalgebra of a real simple Lie algebra other than $\mathfrak{so}^*(4n+2)$, $\mathfrak{e}_{6(-14)}$, $\mathfrak{so}(2n+1-q, q)$ $(n > q > 2)$, and $\mathfrak{sp}(n, n)$ $(n > 1)$. Then, each nontrivial homomorphism between scalar generalized Verma modules induced from $\mathfrak{p}_\Theta$ with regular integral infinitesimal character is composition of elementary homomorphisms.*

For $\mathfrak{so}^*(4n+2)$ and $\mathfrak{e}_{6(-14)}$, we show a weaker statement holds. (cf. 8.3, 8.4)

This article consists of nine sections.

We fix notations and introduce some fundamental material in §1.

In §2, we explain how to reduce the problem to the integral infinitesimal character case. We also show that we can associate an element of $W(\Theta)$ to a homomorphism between generalized Verma modules with regular infinitesimal characters. Finally we formulate the translation principle for generalized Verma modules, which is essentially obtained in [37], [38].

In §3, we introduce the notion of normal parabolic subalgebras and describe the classification. We prove that the Bruhat ordering on $W(\Theta)$ coincides with the restriction of that of W to $W(\Theta)$ for each normal Θ.

In §4, we introduce the notion of an elementary homomorphism and describe related notions and results.

In §5, we introduce the notion of strictly normal parabolic subalgebras and describe the classification. We also prove Theorem B.

In §6, we consider normal but not strictly normal case. We prove that our conjecture is affirmative for a complexified minimal parabolic subalgebras

of $\mathfrak{so}(2n-1,2)$, and normal but strictly normal parabolic subalgebras of exceptional algebras in regular integral infinitesimal character case.

In §7, We consider $\mathfrak{gl}(n,\mathbb{C})$. We introduce the notion of almost normal parabolic subalgebras. We prove that our conjecture is affirmative for an almost normal parabolic subalgebras of $\mathfrak{gl}(n,\mathbb{C})$ in regular integral infinitesimal character case.

In §8, we consider complexified minimal parabolic subalgebras of $\mathfrak{so}^*(4n+2)$ and $\mathfrak{e}_{6(-14)}$.

In §9, we treat two examples.

§ 1. Notations and Preliminaries

1.1 General notations

In this article, we use the following notations and conventions.

As usual we denote the complex number field, the real number field, the ring of (rational) integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. $\frac{1}{2}\mathbb{N}$ means the set $\left\{ \frac{n}{2} \middle| n \in \mathbb{N} \right\}$, and $\frac{1}{2}+\mathbb{N}$ means the set $\left\{ \frac{1}{2}+n \middle| n \in \mathbb{N} \right\}$. We denote by $\emptyset$ the empty set. For any (non-commutative) $\mathbb{C}$-algebra R, "ideal" means "2-sided ideal", "R-module" means "left R-module", and sometimes we denote by 0 (resp. 1) the trivial R-module $\{0\}$ (resp. $\mathbb{C}$). Often, we identify a (small) category and the set of its objects. Hereafter "dim" means the dimension as a complex vector space, and "$\otimes$" (resp. Hom) means the tensor product over $\mathbb{C}$ (resp. the space of $\mathbb{C}$-linear mappings), unless we specify. For a complex vector space V, we denote by V^* the dual vector space. For $a, b \in \mathbb{C}$, "$a \leqslant b$" means that $a, b \in \mathbb{R}$ and $a \leqslant b$. We denote by $A - B$ the set theoretical difference. card$A$ means the cardinality of a set A.

1.2 Notations for reductive Lie algebras

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the root system with respect to $(\mathfrak{g},\mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Let W be the Weyl group of the pair $(\mathfrak{g},\mathfrak{h})$ and let $\langle\ ,\ \rangle$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. For $w \in W$, we denote by $\ell(w)$ the length of w as usual. We also denote the inner product on $\mathfrak{h}^*$ which is induced from the above form by the same symbols $\langle\ ,\ \rangle$. For $\alpha \in \Delta$, we denote by s_α the reflection in W with respect to α. We denote by w_0 the longest element of W. For $\alpha \in \Delta$, we define

the coroot $\alpha^\vee$ by $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$, as usual. We denote by $\Delta^\vee$ the dual root system $\{\alpha^\vee \mid \alpha \in \Delta\}$. We call $\lambda \in \mathfrak{h}^*$ is dominant (resp. anti-dominant), if $\langle\lambda, \alpha^\vee\rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle\lambda, \alpha\rangle \neq 0$, for each $\alpha \in \Delta$. We denote by P the integral weight lattice, namely $\mathsf{P} = \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$. If $\lambda \in \mathfrak{h}^*$ is contained in P, we call λ an integral weight. We define $\rho \in \mathsf{P}$ by $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h} \quad [H,X] = \alpha(H)X\}$, $\mathfrak{u} = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. We denote by Q the root lattice, namely $\mathbb{Z}$-linear span of Δ. We also denote by Q^+ the linear combination of Π with non-negative integral coefficients. For $\lambda \in \mathfrak{h}^*$, we denote by W_λ the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in \mathsf{Q}\}.$$

We denote by Δ_λ the set of integral roots.

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}\}.$$

It is well-known that W_λ is the Weyl group for Δ_λ. We put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. This is a positive system of Δ_λ. We denote by Π_λ the set of simple roots for Δ_λ^+ and denote by S_λ (resp. S) the set of reflection corresponding to the elements in Π_λ (resp. Π). So, (W_λ, S_λ) and (W, S) are Coxeter systems. We denote by Q_λ the integral root lattice, namely $\mathsf{Q}_\lambda = \mathbb{Z}\Delta_\lambda^+$ and put $\mathsf{Q}_\lambda^+ = \mathbb{N}\Pi_\lambda$.

Next, we fix notations for a parabolic subalgebra (which contains $\mathfrak{b}$). Hereafter, through this article we fix an arbitrary subset Θ of Π. Let $\langle\Theta\rangle$ be the set of the elements of Δ which are written by linear combinations of elements of Θ over $\mathbb{Z}$. Put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall\alpha \in \Theta \ \alpha(H) = 0\}$, $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha\in\langle\Theta\rangle}\mathfrak{g}_\alpha$, $\mathfrak{n}_\Theta = \sum_{\alpha\in\Delta^+-\langle\Theta\rangle}\mathfrak{g}_\alpha$, $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$. Then $\mathfrak{p}_\Theta$ is a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$. Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $\Theta \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_\Theta$. We denote by W_Θ the Weyl group for $(\mathfrak{l}_\Theta, \mathfrak{h})$. W_Θ is identified with a subgroup of W generated by $\{s_\alpha \mid \alpha \in \Theta\}$. We denote by w_Θ the longest element of W_Θ. Using the invariant non-degenerate bilinear form $\langle\ ,\ \rangle$, we regard $\mathfrak{a}_\Theta{}^*$ as a subspace of $\mathfrak{h}^*$.

Put $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta\rho)$ and $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta\rho)$. Then, $\rho^\Theta \in \mathfrak{a}_\Theta{}^*$.

For $\Theta \subsetneq \Pi$, we define "the restricted root system" as follows.

$$\Sigma_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Delta\} - \{0\}.$$

$$\Sigma_\Theta^+ = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Delta^+\} - \{0\}.$$

Unfortunately, in general, Σ_Θ does not satisfy the axioms of the root systems.

Define

$$\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta \quad \langle \lambda, \alpha^\vee \rangle \in \{1, 2, ...\}\}$$
$$^\circ\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta \quad \langle \lambda, \alpha^\vee \rangle = 1\}$$

We easily have

$$^\circ\mathsf{P}_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in \mathsf{P}_\Theta^{++}$, we denote by $\sigma_\Theta(\mu)$ the irreducible finite-dimensional $\mathfrak{l}_\Theta$-representation whose highest weight is μ. Let $E_\Theta(\mu)$ be the representation space of $\sigma_\Theta(\mu)$. We define a left action of $\mathfrak{n}_\Theta$ on $E_\Theta(\mu)$ by $X \cdot v = 0$ for all $X \in \mathfrak{n}_\Theta$ and $v \in E_\Theta(\mu)$. So, we regard $E_\Theta(\mu)$ as a $U(\mathfrak{p}_\Theta)$-module.

For $\mu \in \mathsf{P}_\Theta^{++}$, we define a generalized Verma module ([26]) as follows.

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all $\lambda \in \mathfrak{h}^*$, we write $M(\lambda) = M_\emptyset(\lambda)$. $M(\lambda)$ is called a Verma module. For $\mu \in \mathsf{P}_\Theta^{++}$, $M_\Theta(\mu)$ is a quotient module of $M(\mu)$. Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$-module with the highest weight $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in \mathsf{P}_\Theta^{++}$, the canonical projection of $M(\mu)$ to $L(\mu)$ is factored by $M_\Theta(\mu)$.

$\dim E_\Theta(\mu - \rho) = 1$ if and only if $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$. If $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$, we call $M_\Theta(\mu)$ a scalar generalized Verma module.

1.3 Translation functors

We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. It is well-known that $Z(\mathfrak{g})$ acts on $M(\lambda)$ by the Harish-Chandra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ for all λ. $\chi_\lambda = \chi_\mu$ if and only if there exists some $w \in W$ such that $\lambda = w\mu$. We denote by $\mathbf{Z}_\lambda$ the kernel of χ_λ in $Z(\mathfrak{g})$. Let M be a $U(\mathfrak{g})$-module and $\lambda \in \mathfrak{h}^*$. We say that M has an infinitesimal character λ iff $Z(\mathfrak{g})$ acts on M by χ_λ. We say that M has a generalized infinitesimal character λ iff for any $v \in M$ there is some positive integer n such that ${\mathbf{Z}_\lambda}^n v = 0$. We say M is locally $Z(\mathfrak{g})$-finite, iff for any $v \in M$ we have $\dim Z(\mathfrak{g})v < \infty$. We denote by $\mathcal{M}_{Zf}$ (cf [1]) the category of $Z(\mathfrak{g})$-finite $U(\mathfrak{g})$-modules. We also denote by $\mathcal{M}[\lambda]$ the category of $U(\mathfrak{g})$-modules with generalized infinitesimal character λ. Then, from the Chinese remainder theorem, we have a direct sum of abelian categories $\mathcal{M}_{Zf} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{M}[\lambda]$. We denote by P_λ the projection functor from $\mathcal{M}_{Zf}$ to $\mathcal{M}[\lambda]$. For $\mu \in \mathsf{P}$, we denote by V_μ the irreducible finite-dimensional $U(\mathfrak{g})$-module with an extreme weight μ. Let $\mu, \lambda \in \mathfrak{h}^*$

satisfy $\mu - \lambda \in \mathsf{P}$. Let M be an object of $\mathcal{M}[\lambda]$. Then, from a result of Kostant we have that $M \otimes V_{\mu-\lambda}$ is an object of $\mathcal{M}_{Zf}$. So, we can define a translation functor T_λ^μ from $\mathcal{M}[\lambda]$ to $\mathcal{M}[\mu]$ as follows.

$$T_\lambda^\mu(M) = P_\mu(M \otimes V_{\mu-\lambda}).$$

T_λ^μ is an exact functor.

1.4 Submodules of scalar generalized Verma modules

For a finitely generated $U(\mathfrak{g})$-module V, we denote by $\mathrm{Dim}(V)$ the Gelfand-Kirillov dimension of V (cf. [36]).

The following proposition is more or less known.

Proposition 1.4.1. *Let $\Theta \subseteq \Pi$. Then we have :*

(1) Let $\lambda \in \mathsf{P}_\Theta^{++}$. Then, for each nonzero submodule X of $M_\Theta(\lambda)$, we have $Dim(X) = Dim(M_\Theta(\lambda)) = \dim \mathfrak{n}_\Theta$.

(2) Let $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. Then, for each nonzero submodule X of $M_\Theta(\lambda)$, we have $Dim(M_\Theta(\lambda)/X) < Dim(M_\Theta(\lambda)) = \dim \mathfrak{n}_\Theta$.

(3) Let $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. Then $M_\Theta(\lambda)$ has a unique irreducible submodule.

Proof. Since $M_\Theta(\lambda)$ is free of finite rank as a $U(\bar{\mathfrak{n}}_\Theta)$-module, we have $\mathrm{Dim}(M_\Theta(\lambda)) = \dim \bar{\mathfrak{n}}_\Theta = \dim \mathfrak{n}_\Theta$. A nonzero submodule X of $M_\Theta(\lambda)$ is torsion free as $U(\bar{\mathfrak{n}}_\Theta)$-module, so $\dim \bar{\mathfrak{n}}_\Theta \leqslant \mathrm{Dim}(X) \leqslant \mathrm{Dim}(M_\Theta(\lambda)) = \dim \bar{\mathfrak{n}}_\Theta$. So, we have (1). Next, let $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. Then, the multiplicity (the Bernstein degree) (cf. [36]) of $M_\Theta(\lambda)$ is one. So, the number of the irreducible irreducible constituents of $M_\Theta(\lambda)$ which have the Gelfand -Kirillov dimension $\dim \mathfrak{n}_\Theta$ is one. So, from (1), we have (2) and (3). □

§ 2. Formulation of the problem

We retain the notation of §1. In particular, Θ is a proper subset of Π.

2.1 Basic results of Lepowsky

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

Theorem 2.1.1. *([25])*

Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$.

(1) $\dim \mathit{Hom}_{U(\mathfrak{g})}(M_\Theta(\mu), M_\Theta(\nu)) \leqslant 1$.

(2) Any non-zero homomorphism of $M_\Theta(\mu)$ to $M_\Theta(\nu)$ is injective.

Hence, the classification problem of homomorphisms between generalized Verma modules is reduce to the following problem.

Problem 1 Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$. When is $M_\Theta(\mu) \subseteq M_\Theta(\nu)$?

2.2 Reduction to the integral infinitesimal character setting

Since the both $\nu \in W\mu$ and $\nu - \mu \in Q^+$ are necessary conditions for $M_\Theta(\mu) \subseteq M_\Theta(\nu)$, we can reformulate our problem as follows.

Problem 2 Let $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$ be dominant. Let $x, y \in W_\lambda$ be such that $x\lambda, y\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. When is $M_\Theta(x\lambda) \subseteq M_\Theta(y\lambda)$?

We fix $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. Then, we can construct a suralgebra $\mathfrak{g}'$ of $\mathfrak{h}$ such that the corresponding Coxeter system is $(W_\lambda, \Phi_\lambda)$. Since $\Theta \subsetneq \Pi_\lambda$ holds, we can construct the corresponding parabolic subalgebra $\mathfrak{p}'_\Theta$ of $\mathfrak{g}'$. For $\mu \in \mathsf{P}_\Theta^{++}$, we denote by $M'_\Theta(\mu)$ the corresponding generalized Verma module of $\mathfrak{g}'$. We consider the category $\mathcal{O}$ in the sense of [2] corresponding to our particular choice of positive root system. More precisely, we denote by $\mathcal{O}$ (respectively $\mathcal{O}'$) "the category $\mathcal{O}$" for $\mathfrak{g}$ (respectively $\mathfrak{g}'$). We denote by $\mathcal{O}_\lambda$ (respectively, $\mathcal{O}'_\lambda$) the full subcategory of $\mathcal{O}$ (respectively $\mathcal{O}'$) consisting of the objects with a generalized infinitesimal character λ. Soergel's celebrated theorem ([33] Theorem 11) says that there is a Category equivalence between $\mathcal{O}_\lambda$ and $\mathcal{O}'_\lambda$. Under the equivalence a Verma module $M(x\lambda) \quad (x \in W_\lambda)$ corresponds to $M'(x\lambda)$. From Lepowsky's generalized BGG resolutions of the generalized Verma modules and their rigidity, we easily see $M_\Theta(x\lambda)$ corresponds to $M'_\Theta(x\lambda)$ under Soegel's category equivalence. So, we have the following lemma as a corollary of Soergel's theorem.

Lemma 2.2.1. *Let $\lambda \in \mathfrak{h}^*$ be dominant. Let $x, y \in W_\lambda$ be such that $x\lambda, y\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. Then, the following two conditions are equivalent.*

(1) $M_\Theta(x\lambda) \subseteq M_\Theta(y\lambda)$.

(2) $M'_\Theta(x\lambda) \subseteq M'_\Theta(y\lambda)$.

This lemma tells us that we may reduce Problem 2 to the case that λ is integral.

We discuss another application of Soergel's theorem. We denote by $\mathfrak{g}^\vee$ the reductive Lie algebra corresponding to the coroot system $\Delta^\vee$. We regard a Cartan subalgebra $\mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{g}^\vee$. We attach $\vee$ to

the notion with respect to $\mathfrak{g}^\vee$ corresponding to that of $\mathfrak{g}$. Then we have canonical isomorphism $(W, S) \cong (W^\vee, S^\vee)$ of the Coxeter systems. So, we identify them. For $\Theta \subsetneq \Pi$, we put $\Theta^\vee = \{\alpha^\vee \mid \alpha \in \Theta\} \subsetneq \Pi^\vee$. We put ${}^\circ\mathsf{P}^{\vee++}_{\Theta^\vee} = \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \alpha\rangle = 1 \ (\alpha \in \Theta)\}$. For $\lambda \in {}^\circ\mathsf{P}^{\vee++}_{\Theta^\vee}$, we consider a scalar generalized Verma module $M^\vee_{\Theta^\vee}(\lambda)$ of $\mathfrak{g}^\vee$. The following result is an immediate consequence of Soergel's theorem.

Theorem 2.2.2. *Let $\lambda \in \mathsf{P}$ and $\mu \in \mathsf{P}^\vee$ be dominant regular. Let $x, y \in W = W^\vee$. We assume that $x\lambda, y\lambda \in {}^\circ\mathsf{P}^{++}_\Theta$ and $x\mu, y\mu \in {}^\circ\mathsf{P}^{\vee++}_{\Theta^\vee}$. Then, $M_\Theta(x\lambda) \subseteq M_\Theta(y\lambda)$ if and only if $M^\vee_{\Theta^\vee}(x\mu) \subseteq M^\vee_{\Theta^\vee}(y\mu)$.*

Hence, we may reduce Problem 1 for simple Lie algebras of the type C_n to that for simple Lie algebras of the type B_n.

2.3 Comparison of τ-invariants

We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$

Then, we easily have the following lemma.

Lemma 2.3.1. *We have*

(a) $W(\Theta) = \{w \in W \mid w\alpha^\vee \in \Theta^\vee$ *for all* $\alpha \in \Theta.\}$.
(b) $W(\Theta) = \{w \in W \mid w\rho_\Theta = \rho_\Theta, w\Theta \subseteq \Delta^+\}$.
(c) $w_\Theta w = w w_\Theta$ *for all* $w \in W(\Theta)$.
(d) $W(\Theta)$ *preserves* $\mathfrak{a}^*_\Theta$.
(e) $W(\Theta) \subseteq W_{\rho_\Theta}$.

In this section, we prove the following proposition.

Proposition 2.3.2. *Let $\lambda \in {}^\circ\mathsf{P}^{++}_\Theta$ be regular. Let $x \in W_\lambda$ be such that $x\lambda \in {}^\circ\mathsf{P}^{++}_\Theta$. Moreover, we assume that $M_\Theta(x\lambda) \subseteq M_\Theta(\lambda)$. Then, we have $x \in W(\Theta)$.*

First, we prove the following lemma.

Lemma 2.3.3. *Let $\lambda \in {}^\circ\mathsf{P}^{++}_\Theta$ be regular and let $w \in W_\lambda$ be such that $w\lambda$ is dominant. Then, we have $w\Theta \subsetneq \Pi_\lambda$.*

Proof. Assume that there is some $\alpha \in \Theta$ such that $w\alpha \notin \Pi_\lambda$. Then $w\alpha^\vee \notin \Pi^\vee_\lambda$. Here, we remark that $\Pi^\vee_\lambda$ is a basis of the positive coroot system $(\Delta^+)^\vee$. So, there exists some $\beta, \gamma \in \Delta^+$ such that $w\alpha^\vee = \beta^\vee + \gamma^\vee$. Since $w\lambda$ is dominant and regular, we have $\langle w\lambda, \beta^\vee\rangle \geqslant 1$ and $\langle w\lambda, \gamma^\vee\rangle \geqslant 1$. $2 \leqslant \langle w\lambda, \beta^\vee + \gamma^\vee\rangle = \langle w\lambda, w\alpha^\vee\rangle = \langle\lambda, \alpha^\vee\rangle$. On the other hand, $\lambda \in {}^\circ\mathsf{P}^{++}_\Theta$ implies $\langle\lambda, \alpha^\vee\rangle = 1$. This is a contradiction. $\square$

Proof of Proposition 2.3.2

From Lemma 2.2.1, we may reduce the proposition to the case that λ is integral. Put $\Theta_1 = w\Theta$ and $\Theta_2 = wx^{-1}\Theta$. From Lemma 2.3.1, we have $\Theta_1 \subseteq \Pi$ and $\Theta_2 \subseteq \Pi$. Since $w_0 w_{\Theta_i}\Theta_i = -w_0\Theta_i$ holds for $i = 1, 2$, we have $w_0 w_{\Theta_i}\Theta_i \subseteq \Pi$. We put $I_1 = \mathrm{Ann}_{U(\mathfrak{g})}(M_\Theta(\lambda))$ and $I_2 = \mathrm{Ann}_{U(\mathfrak{g})}(M_\Theta(x\lambda))$.

From [7] 4.10 Corollar, we have $I_1 = \mathrm{Ann}_{U(\mathfrak{g})}(M_{-w_0\Theta_1}(w_0 w_{\Theta_1} w\lambda))$ and $I_2 = \mathrm{Ann}_{U(\mathfrak{g})}(M_{-w_0\Theta_2}(w_0 w_{\Theta_2} w\lambda))$. Since $\langle w_0 w_{\Theta_i} w\lambda, \alpha^\vee\rangle < 0$ for all $\alpha \in \Delta^+ - w_0 w_{\Theta_i}\langle\Theta_i\rangle$, $M_{w_0 w_{\Theta_i}\Theta_i}(w_0 w_{\Theta_i} w\lambda)$ is irreducible. Hence, I_1 and I_2 are primitive ideals of the same Gelfand-Kirillov dimension. The τ-invariant of I_1 (respectively I_2) is $-w_0\Theta_1$ (respectively $-w_0\Theta_2$). On the other hand, $M_\Theta(x\lambda) \subseteq M_\Theta(\lambda)$ implies $I_1 \subseteq I_2$. Hence, we have $I_1 = I_2$. Comparing the τ-invariants, we have $-w_0\Theta_1 = -w_0\Theta_2$. Hence, $w\Theta = \Theta_1 = \Theta_2 = wx^{-1}\Theta$. This implies $x \in W(\Theta)$. Q.E.D.

2.4 Translation principle for scalar generalized Verma modules

Next, we consider the images of scalar generalized Verma modules under certain translation functors.

For each $\gamma \in \Sigma_\Theta$, we put$\Delta^\gamma = \{\alpha \in \Delta^+ \mid \alpha|_{\mathfrak{a}_\Theta} = \gamma\}$. We prove:

Lemma 2.4.1. *Let $\gamma \in \Sigma_\Theta$ and let $\beta \in \Delta^\gamma$. If $\langle\rho_\Theta, \beta^\vee\rangle < 0$ (resp. $\langle\rho_\Theta, \beta^\vee\rangle > 0$), then there exists some $\beta' \in \Delta^\gamma$ such that $\langle\rho_\Theta, \beta'^\vee\rangle = \langle\rho_\Theta, \beta^\vee\rangle + 1$ (resp. $\langle\rho_\Theta, \beta'^\vee\rangle = \langle\rho_\Theta, \beta^\vee\rangle - 1$).*

Proof. We assume that $\beta \in \Delta^\gamma$. If $\langle\rho_\Theta, \beta^\vee\rangle < 0$. So, there exists some $\delta \in \Theta$ such that $\langle\delta, \beta^\vee\rangle < 0$. this implies that $\beta^\vee + \delta^\vee \in \Delta^\vee$. Hence there is some $\beta' \in \Delta$ such that $\beta^\vee + \delta^\vee = \beta'^\vee$. This β' satisfies the desirable conditions. The remaining statement is proved in a similar way. □

Lemma 2.4.2. *Let $\Theta \subsetneq \Pi$ and let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is regular integral. Let $\gamma \in \Sigma_\Theta$ be such that $\langle\mu, \gamma\rangle > 0$. Then for each $\beta \in \Delta^\gamma$ we have $\langle\rho_\Theta + \mu, \beta\rangle > 0$.*

Proof. Put $M_\gamma = \{\beta \in \Delta^\gamma \mid \langle\rho_\Theta + \mu, \beta^\vee\rangle < 0\}$. Since $\rho_\Theta + \mu$ is regular, we have only to show $M_\gamma = \emptyset$. Assuming that $M_\gamma \neq \emptyset$, we deduce a contradiction. We choose$\beta_0 \in M_\gamma$ such that $\langle\rho_\Theta + \mu, \beta_0^\vee\rangle$ is maximal among the elements of M_γ. Since $\langle\mu, \beta_0\rangle = \langle\mu, \gamma\rangle > 0$, we may apply Lemma 2.4.1. So, there exists some $\beta' \in \Delta^\gamma$ such that $\langle\rho_\Theta + \mu, \beta'^\vee\rangle = \langle\rho_\Theta + \mu, \beta_0^\vee\rangle + 1$. Since $\rho_\Theta + \mu$ is integral, $\langle\rho_\Theta + \mu, \beta_0^\vee\rangle \leqslant -1$. If $\langle\rho_\Theta + \mu, \beta_0^\vee\rangle < -1$, then $\beta' \in M_\gamma$. It contradicts the choice of β_0. If $\langle\rho_\Theta + \mu, \beta_0^\vee\rangle = -1$, we have $\langle\rho_\Theta + \mu, \beta'^\vee\rangle = 0$. it contradicts our assumption $\rho_\Theta + \mu$ is regular. □

Lemma 2.4.3. *Let $\Theta \subsetneq \Pi$ and let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is regular integral. Then $\rho_\Theta + \mu$ is dominant if and only if $\langle \mu, \gamma \rangle > 0$ for all $\gamma \in \Sigma_\Theta^+$.*

Proof. First, we assume that $\rho_\Theta + \mu$ is dominant. We fix an arbitrary $\gamma \in \Sigma_\Theta^+$. From Lemma 2.4.1, there exists some $\beta \in \Delta^\gamma$ such that $\langle \rho_\Theta, \beta \rangle \geqslant 0$. Then, we have $\langle \mu, \gamma \rangle = \langle \mu, \beta \rangle \geqslant \langle \rho_\Theta + \mu, \beta \rangle > 0$.

Next, we assume that $\langle \mu, \gamma \rangle > 0$ for all $\gamma \in \Sigma_\Theta^+$. From 2.4.2, we see that $\langle \rho_\Theta + \mu, \beta \rangle > 0$ for all $\beta \in \Delta^+ - \mathbb{Z}\Theta$. On the other hand, $\langle \rho_\Theta + \mu, \alpha^\vee \rangle = 1$ for all $\alpha \in \Theta$. So, $\rho_\Theta + \mu$ is dominant. , $\square$

We put $\Sigma_\Theta^+(\nu) = \{\gamma \in \Sigma_\Theta \mid \langle \nu, \gamma \rangle > 0\}$ for $\nu \in \mathfrak{a}_\Theta^*$ such that $\langle \nu, \gamma \rangle \neq 0$ for all $\gamma \in \Sigma_\Theta$.

The following result is immediately deduced from Lemma 2.4.2.

Lemma 2.4.4. *Let $\Theta \subsetneq \Pi$ and let $\mu, \nu \in \mathfrak{a}_\Theta^*$ and $w \in W$ be such that $\rho_\Theta + \mu$ is regular integral and $w(\rho_\Theta + \mu) = \rho_\Theta + \nu$. Then, we have $w = e$ if and only if $\Sigma_\Theta^+(\mu) = \Sigma_\Theta^+(\nu)$.*

We also put

$$K(\Theta) = \{w \in W \mid w\Theta \subseteq \Pi\}.$$

For $\lambda \in \mathfrak{h}^*$ which is regular integral, we put $\Delta^+(\lambda) = \{\alpha \in \Delta \mid \langle \alpha, \lambda \rangle > 0\}$

We consider the following condition (T) on $\mu, \nu \in \mathfrak{a}_\Theta^*$.

Condition (T) $\rho_\Theta + \mu$ and $\rho_\Theta + \nu$ is integral and There exists some $\lambda \in \mathfrak{a}_\Theta^*$ which satisfies the following (T1)-(T3).

(T1) $\rho_\Theta + \lambda$ is regular integral.

(T2) We have $\langle \mu, \gamma \rangle \geqslant 0$ and $\langle \nu, \gamma \rangle \geqslant 0$ for each $\gamma \in \Sigma_\Theta^+(\lambda)$.

(T3) $\mu - \nu$ is dominant with respect to $\Delta^+(\rho_\Theta + \lambda)$.

We fix $\mu, \nu \in \mathfrak{a}_\Theta^*$ satisfying (T). The irreducible finite-dimensional $U(\mathfrak{g})$-module $V = V_{\nu-\mu}$ with a extreme weight $\nu - \mu$ has a filtration $0 = V(0) \subseteq V(1) \subseteq \cdots \subseteq V(k-1) \subseteq V(k) = V$ of $U(\mathfrak{p}_\Theta)$-submodules such that $V(i)/V(i-1) \quad (1 \leqslant i \leqslant k)$ is an irreducible $U(\mathfrak{l}_\Theta)$-modules with the highest weight μ_i. Then, there is some $1 \leqslant i_0 \leqslant k$ such that $\mu_{i_0} = \mu - \nu$ and $\mu_i \neq \mu - \nu$ for all $1 \leqslant i \leqslant k$ such that $i \neq i_0$. We also see that $V(i_0)/V(i_0 - 1)$ is a one-dimensional.

Let $z \in W$ be such that $z(\rho_\Theta + \lambda)$ is dominant. We put $\Theta' = z\Theta$, $\nu' = z\nu$and $\mu' = z\mu$. Then, from Lemma 2.3.3, we have $\Theta' \subseteq \Pi$. Hence $\mu' \in \mathfrak{a}_{\Theta'}^*$ and $z(\rho_\Theta + \mu) = \rho_{\Theta'} + \mu'$. We also put $\mu_i' = z\mu_i$. Twisting by z, $V|_{\lll_{\Theta'}}$ decompose into the direct product of irreducible $\mathfrak{l}_{\Theta'}$-modules with highest weights $\mu_1', ..., \mu_k'$.

The following result is more or less easy consequence of the argument of the proof of [37] Proposition 8.5.

Lemma 2.4.5. ([38] Lemma 4.8, [3] p21 Claim, also see [29] Lemma 1.2.2) *We retain the above settings. Assume $y \in W$. Then, $y(\rho_{\Theta'}+\nu') = \mu_i'+\rho_{\Theta'}+\mu'$ if and only if $y(\rho_\Theta + \nu') = \rho_\Theta + \nu'$ and $i = i_0$.*

When we regard $V = V_{\nu-\mu}$ as a $U(\mathfrak{p}_\Theta)$-module, we write it by $V|_{\mathfrak{p}_\Theta}$. Since $M_\Theta(\rho_\Theta + \mu) \otimes V \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} (E_\Theta(\mu - \rho^\Theta) \otimes V|_{\mathfrak{p}_\Theta})$ holds, we easily see the following from Lemma 2.4.5.

Proposition 2.4.6. *Let $\mu, \nu \in \mathfrak{a}_\Theta^*$ be such that they satisfy the condition (T) above.*

Then, we have

$$T_{\rho_\Theta+\mu}^{\rho_\Theta+\nu}(M_\Theta(\rho_\Theta + \mu)) = M_\Theta(\rho_\Theta + \nu).$$

Finally, we have the following result.

Proposition 2.4.7. *Let $\lambda \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \lambda$ is regular integral and let $w \in W(\Theta)$. We assume that $M_\Theta(\rho_\Theta + w\lambda) \subseteq M_\Theta(\rho_\Theta + \lambda)$. Let $\nu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \nu$ is integral and $\langle \nu, \gamma \rangle \geqslant 0$ for each $\gamma \in \Sigma_\Theta^+(\lambda)$. Then,we have $M_\Theta(\rho_\Theta + w\nu) \subseteq M_\Theta(\rho_\Theta + \nu)$.*

Proof. Since $2\rho_\Theta$ is integral, so is 2λ. Hence for all $k \in \mathbb{N}$, $\rho_\Theta + (2k + 1)\lambda$ is regular integral. It is easy to see $\mu = (2k+1)\lambda$ and ν satisfy the condition (T). From the translation principle, we have $M_\Theta(\rho_\Theta + w\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$. Hence, applying proposition 2.4.7 and the exactness of the translation functor, we have the desired conclusion. □

§ 3. Some results on Bruhat orderings

3.1 Quasi subsystems

Let (W_i, S_i) $(i = 1, 2)$ be finite coxeter systems. We denote by $\ell_i(w)$ the length of a reduced expression of $w \in W_i$ with respect to S_i. We also denote by $\leqslant_i$ the Bruhat ordering for (W_i, S_i).

Definition 3.1.1. *We say that (W_2, S_2) is a quasi subsystem of (W_1, S_1), if the following (Q1) and (Q2) hold.*

(Q1) W_2 is a subgroup of W_1.

(Q2) For any reduced expression $w = s_1 \cdots s_k$ of $w \in W_2$ in (W_2, S_2), we have $\ell_1(w) = \ell_1(s_1) + \cdots + \ell_1(s_k)$.

The following lemma is easy.

Lemma 3.1.2. *Assume that* (W_2, S_2) *is a quasi subsystem of* (W_1, S_1)*. Then,* $x \leqslant_2 y$ *implies* $x \leqslant_1 y$ *for all* $x, y \in W_2$*.*

We have the following lemma.

Lemma 3.1.3. *Assume that* (W_2, S_2) *is a quasi subsystem of* (W_1, S_1)*. Moreover, we assume the following condition (C).*

(C) For any $x, y \in W_2$ *and* $s \in S_2$ *such that* $x \leqslant_1 y$*,* $\ell_1(sy) < \ell_1(y)$*, and* $\ell_1(x) < \ell_1(sx)$*, we have* $sx \leqslant_1 y$*.*

Then, $x \leqslant_1 y$ *implies* $x \leqslant_2 y$ *for all* $x, y \in W_2$*.*

Proof. Let $x, y \in W_2$ be such that $x \leqslant_1 y$. We show $x \leqslant_2 y$ by a double induction with respect to $\ell_2(y)$ and $\ell_2(y) - \ell_2(x)$.

Obviously we may assume $\ell_2(y) > 0$. So, we choose some $s \in S_2$ such that $\ell_2(sy) < \ell_2(y)$.

First, we assume that $\ell_2(sx) < \ell_2(x)$. We fix reduced expressions of s, sx, and sy in (W_1, S_1) as follows.

$$\begin{aligned} s &= s_1 \cdots s_k \qquad (s_1, ..., s_k \in S_1), \\ sx &= t_1 \cdots t_h \qquad (t_1, ..., t_h \in S_1), \\ sy &= r_1 \cdots r_n \qquad (r_1, ..., r_n \in S_1) \end{aligned}$$

From (Q2), we easily see that $s_m \cdots s_k t_1 \cdots t_h$ and $s_m \cdots s_k r_1 \cdots r_n$ are reduced expressions for all $1 \leqslant m \leqslant k$. Applying [10] Theorem 1.1, we have $s_m \cdots s_k t_1 \cdots t_h \leqslant_1 s_m \cdots s_k r_1 \cdots r_n$ by the induction on m. So, we have $sx \leqslant_1 sy$. Since $\ell_2(sy) < \ell_2(y)$, the induction hypothesis implies that $sx \leqslant_2 sy$. Again, applying [10] Theorem 1.1, we have $x \leqslant_2 y$.

Next, we assume that $\ell_2(sx) > \ell_2(x)$. From (Q2), we have $\ell_1(sx) > \ell_1(x)$. So, we have $sx \leqslant_1 y$ from (C). Since $\ell_2(y) - \ell_2(sx) < \ell_2(y) - \ell_2(x)$, we have $sx \leqslant_2 y$ from the induction hypothesis. Since $x \leqslant_2 sx$, we have $x \leqslant_2 y$. ■

3.2 Θ-useful roots

In this subsection, we use the notation in §1.

Following Knapp [20], Howlet [15], and Lusztig [24], we consider useful roots for our purpose.

Hereafter, we fix a subset Θ of Π. For $\alpha \in \Delta$, we put

$$\Delta(\alpha) = \{\beta \in \Delta \mid \exists c \in \mathbb{R} \ \ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\},$$

$$\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+,$$

$$U_\alpha = \mathbb{C}\Theta + \mathbb{C}\alpha \subseteq \mathfrak{h}^*.$$

Then $(U_\alpha, \Delta(\alpha), \langle\ ,\ \rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle\ ,\ \rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. $\alpha|_{\mathfrak{a}_\Theta} = 0$ if and only if $\Theta = \Pi(\alpha)$. For $\alpha \in \Delta^+$, we denote by $W_\Theta(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$. We denote by w^α the longest element of $W_\Theta(\alpha)$. We put as follows.

$$\sigma_\alpha = w^\alpha w_\Theta.$$

$\alpha|_{\mathfrak{a}_\Theta} = 0$ if and only if $\sigma_\alpha = 1$. If $\alpha \in \Delta$ is orthogonal to all the elements in Θ, then we can easily see α is Θ-reduced and $s_\alpha = \sigma_\alpha$.

Definition 3.2.1. (1) We call $\alpha \in \Delta$ Θ-*useful* if the order of σ_α is two. We denote by ${}^u\Delta_\Theta$ the set of the useful Θ-roots. We also put ${}^u\Delta_\Theta^+ = {}^u\Delta_\Theta \cap \Delta^+$.

(2) If $\alpha|_{\mathfrak{a}_\Theta} \neq 0$, then $\Pi(\alpha)$ is written as $S \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call α Θ-*reduced.* We put

$$ {}^{ru}\Delta_\Theta^+ = \{\alpha \in {}^u\Delta_\Theta^+ \mid \alpha \textit{ is } \Theta\textit{-reduced.}\}$$

We easily see:

Lemma 3.2.2. *Let $\alpha \in \Delta^+$ be Θ-reduced. We denote by $\Delta(\alpha)_0$ be the irreducible component of $\Delta(\alpha)$ containing α. We put $\Pi(\alpha)_0 = \Pi(\alpha) \cap \Delta(\alpha)_0$.*

(1) If $\Delta(\alpha)_0$ is not of the type ADE, then we have $\alpha \in {}^{ur}\Delta_\Theta^+$.

(2) If $\Delta(\alpha)_0$ is of the type D_{2n} $(n \geqslant 2)$, E_7, or E_8, then we have $\alpha \in {}^{ur}\Delta_\Theta^+$.

(3) If $\Delta(\alpha)_0$ is of the type A_{2n} $(n \geqslant 1)$, then we have $\alpha \notin {}^{ur}\Delta_\Theta^+$.

(4) We assume that $\Delta(\alpha)_0$ is of the type A_{2n+1} $(n \geqslant 0)$. We number the elements of $\Pi(\alpha)_0$ as follows.

$$\Pi(\alpha)_0 = \{\beta_1, ..., \beta_{2n+1}\}.$$

We choose the above numbering so that $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leqslant i \leqslant 2n$. Then $\alpha \in {}^{ur}\Delta_\Theta^+$ if and only if $\alpha = \beta_n$.

(5) We assume that $\Delta(\alpha)_0$ is of the type D_{2n+1} $(n \geqslant 2)$. We number the elements of $\Pi(\alpha)_0$ as follows.

$$\Pi(\alpha)_0 = \{\beta_1, ..., \beta_{2n+1}\}.$$

We choose the above numbering so that $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leqslant i \leqslant 2n-1$ and $\langle \beta_{2n-1}, \beta_{2n+1}^\vee \rangle = -1$. Then $\alpha \in {}^{ur}\Delta_\Theta^+$ if and only if $\alpha \notin \{\beta_{2n}, \beta_{2n+1}\}$.

(6) We assume that $\Delta(\alpha)_0$ is of the type E_6. We number the elements of $\Pi(\alpha)_0$ as follows.

$$\Pi(\alpha)_0 = \{\beta_1, ..., \beta_6\}.$$

We choose the above numbering so that $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leqslant i \leqslant 4$ and $\langle \beta_3, \beta_6^\vee \rangle = -1$. Then $\alpha \in {}^{ur}\Delta_\Theta^+$ if and only if $\alpha \in \{\beta_3, \beta_6\}$.

For $\alpha \in {}^{ru}\Delta_\Theta$, we put

$$V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha \rangle = 0\},$$

$$\hat{\alpha} = \alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^*.$$

We easily see:

Lemma 3.2.3. *Let $\alpha \in {}^{ru}\Delta_\Theta^+$. Then, we have*

(1) σ_α preserves $\mathfrak{a}_\Theta^*$.

(2) $\sigma_\alpha \in W(\Theta)$. In particular, $\sigma_\alpha \rho_\Theta = \rho_\Theta$.

(3) $\sigma_\alpha \hat{\alpha} = -\hat{\alpha}$.

(4) $\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$ is the reflection with respect to V_α.

We denote by $W(\Theta)'$ the subgroup of W generated by $\{\sigma_\alpha \mid \alpha \in {}^{ru}\Delta_\Theta^+\}$. We put ${}^u\Sigma_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^* \mid \alpha \in {}^u\Delta_\Theta\}$. ${}^u\Sigma_\Theta$ is a (not necessarily reduced) root system. We also put ${}^{ru}\Sigma_\Theta^+ = \{\alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^* \mid \alpha \in {}^{ru}\Delta_\Theta^+\}$ and ${}^{ru}\Sigma_\Theta = {}^{ru}\Sigma_\Theta^+ \cup -{}^{ru}\Sigma_\Theta^+$. ${}^{ru}\Sigma_\Theta$ is a reduced root system and ${}^{ru}\Sigma_\Theta^+$ is a positive system. We denote by ${}^u\Pi_\Theta$ the simple system for ${}^{ru}\Sigma_\Theta^+$. ${}^u\Pi_\Theta$ is also a basis of ${}^u\Sigma_\Theta$. For $\alpha \in {}^{ru}\Delta_\Theta^+$, σ_α depends only on $\alpha|_{\mathfrak{a}_\Theta}$. So, sometimes we write $\sigma_{\alpha|_{\mathfrak{a}_\Theta}}$ for σ_α. We put $S(\Theta) = \{\sigma_\gamma \mid \gamma \in {}^u\Pi_\Theta\}$

Theorem 3.2.4. (Howlet [15] Theorem 6, Lusztig [24] §5)

(1) $W(\Theta)' \subseteq W(\Theta)$.

(2) For $\alpha \in {}^u\Delta_\Theta^+$, $\sigma_\alpha(\mathfrak{a}_\Theta^*) = \mathfrak{a}_\Theta^*$. Moreover, $\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$ is the reflection with respect to $\alpha|_{\mathfrak{a}_\Theta}$ and $\sigma_\alpha \rho_\Theta = \rho_\Theta$.

(3) We define $\iota : W(\Theta)' \to GL(\mathfrak{a}_\Theta^*)$ by $\iota(x) = x|_{\mathfrak{a}_\Theta^*}$. Then ι is an injective group homomorphism.

(4) $\iota(W(\Theta)')$ is the reflection group for the root system ${}^{ru}\Sigma_\Theta$. Hence $(W(\Theta)', S(\Theta))$ is a Coxeter system.

We denote by $\leqslant_\Theta$ the Bruhat ordering for $(W(\Theta)', S(\Theta))$.

3.3 Normal parabolic subalgebras

Definition 3.3.1. *We call $\Theta \subsetneq \Pi$ normal, if $\Pi - \Theta \subseteq {}^u\Delta^+_\Theta$. We call a standard parabolic subalgebra $\mathfrak{p}_\Theta$ normal, if Θ is normal. A parabolic subalgebra is called normal, if it is conjugate to a normal standard parabolic subalgebra by an inner automorphism.*

We describe the list of the normal parabolic subalgebras of classical Lie algebras.

(1) Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is similar) and let k be a positive integer dividing n. We consider the following parabolic subalgebras.

$\mathfrak{p}(A_{n-1,k})$: a parabolic subalgebra of $\mathfrak{g}$ whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{n/k}.$$

(2) Let $\mathfrak{g}$ be a complex simple Lie algebra of the type X_n. Here, X means one of B, C, and D. Let k and ℓ be positive integers such that k divides $n - \ell$. If $X = D$, then we assume that $\ell \neq 1$.

We consider the following parabolic subalgebras.

$\mathfrak{p}(X_{n,k,\ell})$: a parabolic subalgebra of $\mathfrak{g}$ whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{(n-\ell)/k} \oplus X_\ell.$$

Here, X_ℓ means that the complex simple Lie algebra of the type X_ℓ. Namely $B_\ell = \mathfrak{so}(2\ell + 1, \mathbb{C})$, $C_n = \mathfrak{sp}(\ell, \mathbb{C})$, and $D_n = \mathfrak{so}(2\ell, \mathbb{C})$. X_0 means the zero Lie algebra.

From lemma 3.2.2, we easily see:

Proposition 3.3.2. *(1) $\mathfrak{p}(A_{n-1,k})$ is normal. Conversely any normal parabolic subalgebra of is conjugate to $\mathfrak{p}(A_{n,k})$ for some k.*

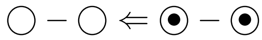
(2) $\mathfrak{p}(X_{n,k,\ell})$ is normal $X = D$, $\ell = 0$, and k is an odd number greater than 1. Any normal parabolic subalgebra is conjugate to one of such $\mathfrak{p}(X_{n,k,\ell})$s by an inner automorphism.

For exceptional simple Lie algebras, we have the following results. We describe $\Theta \subsetneq \Pi$ by a marked Dynkin diagram. We write "$\odot$" the vertices corresponding to elements in Θ. If Θ is the empty set, it is obviously normal. So, we consider $\emptyset \neq \Theta \subsetneq \Pi$.

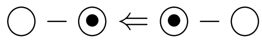
Proposition 3.3.3. *(1) Assume that $\mathfrak{g}$ is of type G_2. Then any subset of Π is normal.*

(2) Assume that $\mathfrak{g}$ is of type F_4*. If card*$\Theta = 3$*,* $\Theta \subsetneq \Pi$ *is normal. The list of the other nonempty normal subsets of* Π *is as follows.*

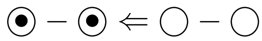
$\mathrm{F}_{4,12}$



$\mathrm{F}_{4,14}$

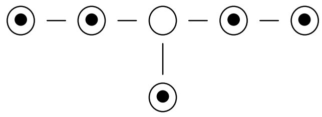


$\mathrm{F}_{4,34}$

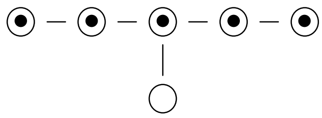


(3) Assume that $\mathfrak{g}$ is of type E_6*. Then the list of the nonempty normal subsets of* Π *is as follows.*

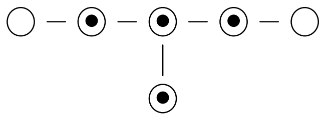
$\mathrm{E}_{6,3}$



$\mathrm{E}_{6,6}$

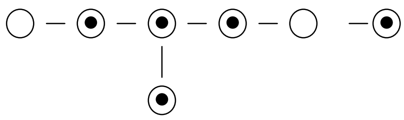


$\mathrm{E}_{6,15}$

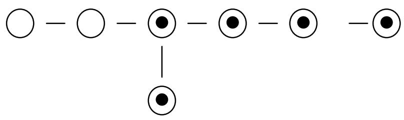


(4) Assume that $\mathfrak{g}$ is of type E_7*. If card*$\Theta = 6$*,* $\Theta \subsetneq \Pi$ *is normal. The list of the other nonempty normal subsets of* Π *is as follows.*

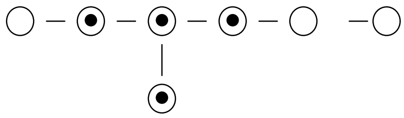
$\mathrm{E}_{7,27}$



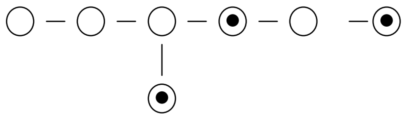
$\mathrm{E}_{7,67}$



$E_{7,127}$

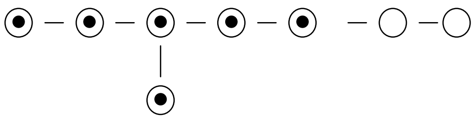


$E_{7,2467}$

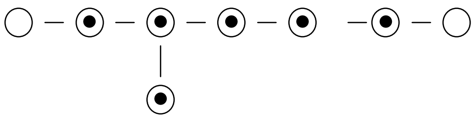


(5) Assume that $\mathfrak{g}$ *is of type* E_8. *If card*$\Theta = 7$, $\Theta \subsetneq \Pi$ *is normal. The list of the other nonempty normal subsets of* Π *is as follows.*

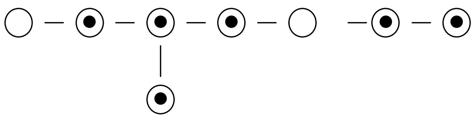
$E_{8,12}$



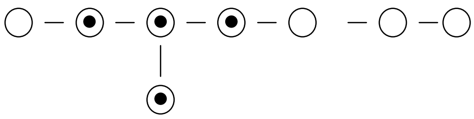
$E_{8,18}$



$E_{8,38}$



$E_{8,1238}$



We give some characterization of normality.

Proposition 3.3.4. *For* $\Theta \subsetneq \Pi$. *the following conditions are equivalent.*

(1) $\Theta \subsetneq \Pi$ *is normal.*

(2) $K(\Theta) = W(\Theta)'$.

(3) $K(\Theta) = W(\Theta)$.

(4) ${}^u\Sigma_\Theta = \Sigma_\Theta$

Proof. First, we assume (1) . Then, using Proposition 3.3.2 and 3.3.3, we obtain (2) and (4) via case-by-case analysis. (2) obviously implies (3). Next, we assume (3). For $\alpha \in \Pi - \Theta$, we easily see $\sigma_\alpha^2(\Pi) \subseteq \Delta^+$. Hence σ_α is an involution. This means that $\alpha \in {}^u\Delta^+_\Theta$. So, we have (1). (4) is clearly stronger than (1). $\square$

Corollary 3.3.5. *If $\Theta \subsetneq \Pi$ is normal, then $W(\Theta)' = W(\Theta)$.*

Since $\Delta^+ \cap (-w\Delta^+) = \{\alpha \in \Delta^+ \mid \alpha|_{\mathfrak{a}_\Theta} \in \Sigma^+_\Theta \cap (-w\Sigma^+_\Theta)\}$ for each $w \in W(\Theta)$, we easily see the following lemma.

Lemma 3.3.6. *We assume that $\Theta \subsetneq \Pi$ is normal. Then for each $w \in W(\Theta)$, we have*

$$\Delta^+ \cap (-w\Delta^+) = \bigcup_{\gamma \in {}^{ru}\Sigma^+_\Theta \cap (-w\,{}^{ru}\Sigma^+_\Theta)} \{\alpha \in \Delta^+ \mid \exists c > 0 \ \ \alpha|_{\mathfrak{a}_\Theta} = c\gamma\}.$$

Hence, we have the following result.

Proposition 3.3.7. *If $\Theta \subsetneq \Pi$ is normal, then $(W(\Theta)', S(\Theta))$ is a quasi subsystem of (W, S).*

As a corollary of Proposition 3.3.4, we easily have:

Corollary 3.3.8. *$\Theta \subsetneq \Pi$ is normal if and only if any two parabolic subalgebras with the Levi part $\mathfrak{l}_\Theta$ are conjugate under an inner automorphism of $\mathfrak{g}$.*

3.4 Comparison of Bruhat orderings

In this subsection, we use the notation in §1.

Definition 3.4.1. We call $\Theta \subsetneq \Pi$ *seminormal*, if there exists some Ψ such that $\Theta \subseteq \Psi \subseteq \Pi$ and ${}^u\Pi_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Psi - \Theta\}$.

So, $S(\Theta) = \{\sigma_\alpha \mid \alpha \in \Theta - \Psi\}$.

$\Theta \subsetneq \Pi$ is seminormal if and only if there is a $\alpha \in \Pi \cap {}^{ru}\Delta^+$ such that $\alpha|_{\mathfrak{a}_\Theta} = \gamma$ for each $\gamma \in {}^u\Pi_\Theta$.

We immediately see the following result from Proposition 3.3.7.

Corollary 3.4.2. *If $\Theta \subsetneq \Pi$ is seminormal, then $(W(\Theta)', S(\Theta))$ is a quasi subsystem of (W, S).*

We fix a connected complex reductive Lie group G whose Lie algebra is $\mathfrak{g}$. For $\Theta \subsetneq \Pi$, we denote by P_Θ (resp. H) the parabolic subgroup (resp. the Cartan subgroup) of G corresponding to $\mathfrak{p}_\Theta$ (resp. $\mathfrak{h}$). We denote by $N_G(H)$ the normalizer of H in G. Since the Weyl group W is identified with the

quotient group $N_G(H)/H$, for each $w \in W$ we can fix a representative in $N_G(H)$. We denote the representative by the same letter "w".

For $x \in W$, we put $U_x = P_\Theta x/P_\Theta$. Namely, U_x is a P_Θ-orbit in G/P_Θ through $x/P_\Theta \in G/P_\Theta$. We denote by $\overline{U}_x$ the closure of U_x in G/P_Θ. If $w \in W(\Theta)$, then $\ell(ws_\alpha) > \ell(w)$ for all $\alpha \in \Theta$. Hence, we have

Lemma 3.4.3. *(1) For $w \in W(\Theta)$, we have* $\dim U_w = \ell(w)$.

(2) For $x, y \in W(\Theta)$, $x \leqslant y$ if and only if $\overline{U}_x \subseteq \overline{U}_y$.

Next we show,

Lemma 3.4.4. *Assume that $\Theta \subsetneq \Pi$ is seminormal. We choose $\Theta \subseteq \Psi \subseteq \Pi$ as in 3.4.1. Fix $x \in W(\Theta)'$. Let $\alpha \in \Psi - \Theta$ be such that $\ell(\sigma_\alpha x) < \ell(x)$. Then we have $\overline{U}_x = P_{\Theta\cup\{\alpha\}}\overline{U}_x = P_{\Theta\cup\{\alpha\}}\overline{U}_{\sigma_\alpha x}$.*

Proof. We may choose a reduced expression $x = \sigma_{\alpha_1}\cdots\sigma_{\alpha_k}$ such that $\alpha_1 = \alpha$. We consider a contraction map as follows.

$$F : P_{\Theta\cup\{\alpha_1\}} \times_{P_\Theta} P_{\Theta\cup\{\alpha_2\}} \times_{P_\Theta} \cdots \times_{P_\Theta} P_{\Theta\cup\{\alpha_k\}}/P_\Theta \to G/P_\Theta.$$

We easily see :

(a) Image(F) is an irreducible Zariski closed set in G/P_Θ.

(b) $\dim \overline{U}_x = \ell(x) = \dim P_{\Theta\cup\{\alpha_1\}} \times_{P_\Theta} \cdots \times_{P_\Theta} P_{\Theta\cup\{\alpha_k\}}/P_\Theta$.

(c) $\overline{U}_x \subseteq$ Image(F).

Hence, we have $\overline{U}_x =$ Image(F). So, we have the lemma immediately. □

The following result is the main result of this section.

Theorem 3.4.5. *Let $\Theta \subsetneq \Pi$ be seminormal. For $x, y \in W(\Theta)'$, $x \leqslant y$ if and only if $x \leqslant_\Theta y$.*

Proof. We choose $\Theta \subseteq \Psi \subseteq \Pi$ as in 3.4.1. From Lemma 3,1.2, Lemma 3.1.3, and Corollary 3.4.2, we have only to show the condition (C) in the statement of Lemma 3.1.3 holds for $(W(\Theta)', S(\Theta))$. So we choose $x, y \in W(\Theta)'$ and $\alpha \in \Psi - \Theta$ such that $x \leqslant y$, $\ell(\sigma_\alpha y) < \ell(y)$, and $\ell(\sigma_\alpha x) > \ell(x)$. From $x \leqslant y$, we have $\overline{U}_x \subseteq \overline{U}_y$ by Lemma 3.4.3 (2). Hence $P_{\Theta\cup\{\alpha\}}\overline{U}_x \subseteq P_{\Theta\cup\{\alpha\}}\overline{U}_y$. From Lemma3.4.4, we have $\overline{U}_y = P_{\Theta\cup\{\alpha\}}\overline{U}_y$ and $\overline{U}_{\sigma_\alpha x} = P_{\Theta\cup\{\alpha\}}\overline{U}_x$. So, we have $\overline{U}_{\sigma_\alpha x} \subseteq \overline{U}_y$. this means that $\sigma_\alpha x \leqslant y$. Hence, the condition (C) holds for Θ. Q.E.D.

§ 4. Elementary homomorphisms

4.1 Elementary homomorphisms

We fix a subset Θ of Π and $\alpha \in {}^{ru}\Delta_\Theta^+$. We define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta\in\Delta(\alpha)} \mathfrak{g}_\beta, \qquad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of $\mathfrak{g}$ whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

We denote by $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ the fundamental weight for α with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely ω_α satisfies that $\langle \omega_\alpha, \beta\rangle = 0$ for $\beta \in \Theta$, $\langle \beta, \alpha^\vee\rangle = 1$, and $\omega_\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number a such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta}$, since $\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle\lambda, \omega_\alpha\rangle = 0\}$.

Put $\rho(\alpha) = \frac{1}{2}\sum_{\beta\in\Delta^+(\alpha)}\beta$, For $\nu \in \mathfrak{a}_\Theta^*$, we denote by $\mathbb{C}_\nu$ the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$-module corresponding to ν. For $\nu \in \mathfrak{a}_\Theta^*$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows.

$$M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu-\rho(\alpha)}.$$

Then, we have:

Theorem 4.1.1. *([30]) Let ν be an arbitrary element in V_α, let c be either 1 or $\frac{1}{2}$. Assume that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - c\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + c\omega_\alpha)$. Then, we have $M_\Theta(\rho_\Theta + \nu - (c+n)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + (c+n)\omega_\alpha)$ for all $n \in \mathbb{N}$.*

We call the above homomorphism of $M_\Theta(\rho_\Theta+\nu-(c+n)\omega_\alpha)$ into $M_\Theta(\rho_\Theta+\nu+(c+n)\omega_\alpha)$ *an elementary homomorphism.* In [30], homomorphisms between scalar generalized Verma modules associated with a maximal parabolic subalgebra are classified. So, elementary homomorphisms are understood.

The following conjecture is propsed in [30] as a working hypothesis.

Conjecture 4.1.2. *An arbitrary nontrivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.*

The conjecture in the case of the Verma modules is nothing but the result of Bernstein-Gelfand-Gelfand ([2]). I do not know a counterexample for the above working hypothesis and we obtain partial affirmative results in this article. A weaker version is :

Conjecture 4.1.3. *Let $\Theta \subseteq \Pi$ be normal and let $\mu, \nu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ and $\rho_\Theta + \nu$ are regular integral. If $M_\Theta(\rho_\Theta + \nu) \subseteq M_\Theta(\rho_\Theta + \mu)$, then it is a composition of elementary homomorphisms.*

Later, we show that the conjecture is affirmative for strictly normal case (see §5) and exceptional Lie algebras (see §5 and §6).

For example, I do not know whether an homomorphism of the form $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$ $(\mu \in \mathfrak{a}_\Theta^*)$ is always elementary. We have a weak result.

Proposition 4.1.4. *We fix $\mu \in \mathfrak{a}_\Theta^*$ such that $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$ and $\rho_\Theta + \mu$ is regular and integral. If $\{\beta \in \Sigma_\Theta - \mathbb{R}\alpha|_{\mathfrak{a}_\Theta} \mid \langle\mu, \beta\rangle > 0\} = \{\beta \in \Sigma_\Theta - \mathbb{R}\alpha|_{\mathfrak{a}_\Theta} \mid \langle\sigma_\alpha\mu, \beta\rangle > 0\}$, then $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$ is an elementary homomorphism.*

Proof. Put $\nu_0 = \mu - \langle\mu, \alpha^\vee\rangle\omega_\alpha$. Then $\nu_0 \in V_\alpha$. Since $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$, we have $\mu - \sigma_\alpha\mu = 2\langle\mu, \alpha^\vee\rangle\omega_\alpha \in \mathsf{Q}^+$. Hence, $2\langle\mu, \alpha^\vee\rangle\omega_\alpha$ is integral. So, we can write $\langle\mu, \alpha^\vee\rangle = c + n_0$. Here, c is either 1 or $\frac{1}{2}$ and n_0 is a positive integer. Put $\kappa = 2(\mu + \sigma_\alpha\mu)$ Since $2\rho_\Theta$ and $\rho_\Theta + \mu$ are integral, so is κ. Moreover, we have $\kappa \in V_\alpha$ and $\langle\kappa, \beta\rangle > 0$ for all $\beta \in \Sigma_\Theta - \mathbb{R}\alpha|_{\mathfrak{a}_\Theta}$ such that $\langle\mu, \beta\rangle > 0$. From the translation principle, we have $M_\Theta(\rho_\Theta + (\nu_0 + m\kappa) - (c + n_0)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + (\nu_0 + m\kappa) + (c + n_0)\omega_\alpha)$ for all $m \in \mathbb{N}$. Hence $\{a \in \mathbb{C} \mid M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) - (c + n_0)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) + (c + n_0)\omega_\alpha)\}$ is Zariski dense in $\mathbb{C}$. So, we can prove $M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) - (c + n_0)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) + (c + n_0)\omega_\alpha)$ for all $a \in \mathbb{C}$ in the same way as [24] Lemma 5.4. If $a \in \mathbb{C}$ is generic, then the integral toot system for $(\rho_\Theta + (\nu_0 + a\kappa) - (c+n_0)\omega_\alpha$ is $\Delta(\alpha)$. Hence, Lemma 2.2.1 implies that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - (c+n_0)\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + (c+n_0)\omega_\alpha)$. Applying [30] Lemma 2.2.6, we have $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - c\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + c\omega_\alpha)$. □

4.2 Θ-excellent roots

We retain the notations in 4.1.

Definition 4.2.1. (1) We call $\alpha \in {}^{ru}\Delta = \Theta^+$ Θ-excellent if σ_α is a Duflo involution ([11] cf. [19]) in $W(\alpha)$.

(2) We put ${}^e\Delta_\Theta^+ = \{\alpha \in {}^{ru}\Delta_\Theta^+ \mid \alpha \text{ is } \Theta\text{-excellent}\}$.

(3) We put ${}^e\Sigma_\Theta^+ = \{\alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^* \mid \alpha \in {}^e\Delta_\Theta^+\}$ and ${}^e\Sigma_\Theta = {}^e\Sigma_\Theta^+ \cup (-{}^e\Sigma_\Theta^+)$.

(4) We denote by ${}^eW(\Theta)$ the subgroup of $W(\Theta)'$ generated by $\{\sigma_\alpha \mid \alpha \in {}^e\Delta_\Theta^+\}$.

(5) For $\alpha \in {}^{ru}\Delta_\Theta^+$, we put $c_\alpha = 1$ (resp. $c_\alpha = \frac{1}{2}$) if ρ_Θ is integral (resp. not integral) with respect to $\Delta(\alpha)$. Then, $\rho_\Theta + (c_\alpha + n)\omega_\alpha$ is integral with respect to $\Delta(\alpha)$ for all $n \in \mathbb{Z}$.

We have

Proposition 4.2.2. *Let $\alpha \in {}^e\Delta_\Theta^+$ and let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is integral and $\langle \mu, \alpha \rangle > 0$. Then, we have an elementary homomorphism $M_\Theta(\rho_\Theta + \sigma_\alpha \mu) \subseteq M_\Theta(\rho_\Theta + \mu)$.*

Proof. Put $\nu_0 = \mu - \langle \mu, \alpha^\vee \rangle \omega_\alpha$. Then $\nu_0 \in V_\alpha$. Since $\rho_\Theta + \mu$ is integral, we have $\langle \rho_\Theta + \mu, \alpha^\vee \rangle \in \mathbb{Z}$. From he definition of c_α, we have $\langle \rho_\Theta, \alpha^\vee \rangle \in c_\alpha + \mathbb{Z}$. Hence, we can write $\mu = \nu_0 + (c_\alpha + n)\omega_\alpha$ for some $n \in \mathbb{N}$. So, from $\alpha \in {}^e\Delta_\Theta^+$, Theorem 4.1.1, Proposition 2.4.7, and [29] Proposition 2.1.2, we have the proposition. $\square$

For a simple Lie algebra of the type A, every involution is a Duflo involution ([?]). hence, we have:

Corollary 4.2.3. *If $\mathfrak{g}$ is a simple Lie algebra of the type A, we have ${}^{ru}\Delta_\Theta^+ = {}^e\Delta_\Theta^+$ for all $\Theta \subsetneq \Pi$.*

§ 5. Strictly normal case

5.1 Strictly normal subset of Π

Definition 5.1.1. We call $\Theta \subsetneq \Pi$ *strictly normal*, if Θ is normal and ${}^e\Delta_\Theta^+ = {}^{ru}\Delta_\Theta^+$. A standard parabolic subalgebra $\mathfrak{p}_\Theta$ is called strictly normal when Θ is strictly normal.

Before stating the main result, we prove the following lemma.

Lemma 5.1.2. *Let $\Theta \subsetneq \Pi$ be normal and let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is integral. Then, μ is integral with respect to ${}^{ru}\Sigma_\Theta$.*

Proof. Since $\rho_\Theta + \mu$ is integral, we have $w(\rho_\Theta + \mu) - \rho_\Theta - mu = w\mu - \mu \in \mathsf{Q}$ for all $w \in W(\Theta)'$. Since $\mathsf{Q} \cap \mathfrak{a}_\Theta^*$ is contained in the root lattice for ${}^{ru}\Sigma_\Theta$, we have the result. $\square$

The following result is the main result.

Theorem 5.1.3. *We assume that $\Theta \subsetneq \Pi$ is strictly normal. Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is dominant integral and regular. Let $x, y \in W(\Theta)'$. Then, we have*

(1) $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ if and only if $y \leqslant_\Theta x$.

(2) If $y \leqslant_\Theta x$, then $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ is a composition of elementary homomorphisms.

Proof. First, we assume that $M_\Theta(\rho_\Theta+x\mu) \subseteq M_\Theta(\rho_\Theta+y\mu)$. Hence, $L(\rho_\Theta+x\mu)$ is an irreducible constituent of $M(\rho_\Theta+y\mu)$. From [2], we have$M(\rho_\Theta+x\mu) \subseteq M(\rho_\Theta+y\mu)$, namely $y \leqslant x$. Hence from Theorem 3.4.5, we have $y \leqslant_\Theta x$.

Next, we assume that $y \leqslant_\Theta x$. Since μ is regular domimant integral with respect to ${}^{ru}\Sigma_\Theta$, there exist $\alpha_1, ..., \alpha_\in {}^{ru}\Delta_\Theta^+$ such that $\sigma_{\alpha_1}\cdots\sigma_{\alpha_k}y = x$, $\langle y\mu, \alpha_k\rangle > 0$, and $\langle \sigma_{\alpha_{r+1}}\cdots\sigma_{\alpha_k}y\mu, \alpha_r\rangle > 0$ for $1 \leqslant r \leqslant k-1$. So, from Proposition 4.2.2, we can construct embedding $M_\Theta(\rho_\Theta+x\mu) \subseteq M_\Theta(\rho_\Theta+y\mu)$ as a composition of elementary homomorphisms.

Q.E.D.

5.2 Classification of the strictly normal parabolic subalgebras

From [30], we can determine Θ-excellent roots and we can obtain the following result.

Proposition 5.2.1. *The following is the list of the strictly normal standard parabolic subalgebras of a classical Lie algebra.*

(a) $\mathfrak{p}(A_{n-1,k})$ $(k|n)$,

(b) $\mathfrak{p}(B_{n,2k,m})$ $(k \leqslant m)$,

(c) $\mathfrak{p}(B_{n,2k+1,m})$ $(k \geqslant m)$,

(d) $\mathfrak{p}(C_{n,2k,m})$ $(k \leqslant m)$,

(e) $\mathfrak{p}(C_{n,2k+1,m})$ $(k \geqslant m)$,

(f) $\mathfrak{p}(D_{n,2k-1,m})$ $(k \leqslant m,\ 2 \leqslant m)$,

(g) $\mathfrak{p}(D_{n,2k,m})$ $(k \geqslant m, 2 \leqslant m\ )$,

(h) $\mathfrak{p}(D_{n,1,0})$.

Next, we state the classification of strictly normal parabolic subalgebras for exceptional Lie algebras. It is obtained by more or less straightforward calculation from [30]. .

Proposition 5.2.2. *Let $\mathfrak{g}$ be an exceptional Lie algebra. We assume $\Theta \subsetneq \Pi$ is normal and* $\mathrm{card}(\Pi-\Theta) \geqslant 2$. *Moreover, we assume that Θ is not strictly normal. Then Θ is $F_{4,14}$, $E_{7,27}$, or $E_{8,18}$.*

Remark If $\mathrm{card}(\Pi-\Theta) = 1$, then $\mathfrak{p}_\Theta$ is a maximal parabolic subalgebra. In this case, the homomorphisms between scalar generalized Verma modules are classified in [30]. So, we neglect them.

§ 6. Normal but not strictly normal case

6.1 General results

We assume that $\mathfrak{g}$ is simple and $\Theta \subsetneq \Pi$ is normal but not strictly normal. We also assume $\mathfrak{p}_\Theta$ is not a maximal parabolic subalgebra, namely $\mathrm{card}(\Pi - \Theta) \geqslant 2$. Then, we easily see from the classification that ${}^{ru}\Sigma_\Theta$ is of the type B_n. Here, $n = \dim \mathfrak{a}_\Theta$. Moreover, we assume Θ is not of the type $F_{4,14}$. Then, ${}^{e}\Sigma_\Theta$ is the set of the long roots in ${}^{ru}\Sigma_\Theta$. So, ${}^{e}\Sigma_\Theta$ is a root system of the type D_n. We put ${}^{ru}\Pi_\Theta = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ such that $\langle \gamma_i, \gamma_{i+1}^\vee \rangle = -1$ for $1 \leqslant i < n$ and $\langle \gamma_2, \gamma_1^\vee \rangle = -2$. Namely, γ_1 is a unique short simple root. Put $\gamma' = \sigma_{\gamma_1}\gamma_2$. Then, $\{\gamma', \gamma_2, ..., \gamma_n\}$ is a basis of ${}^{e}\Sigma_\Theta^+$. We put ${}^{e}S(\Theta) = \{\sigma_{\gamma'}\} \cup \{\sigma_{\gamma_i} \mid 2 \leqslant i \leqslant n\}$ and we denote by $\leqslant'_\Theta$ the Bruhat ordering for a Coxeter system $({}^{e}W(\Theta), {}^{e}S(\Theta))$. We remark that the index of ${}^{e}W(\Theta)$ in $W(\Theta)'$ is two and $W(\Theta)' = {}^{e}W(\Theta) \cup {}^{e}W(\Theta)\sigma_{\gamma_1}$.

The argument of the proof of Theorem 5.1.5 is partially applicable in this setting and we have the following weaker result.

Proposition 6.1.1. *We retain the above setting. Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is dominant integral and regular.*

(1) Let $x, y \in {}^{e}W(\Theta)$ be such that $y \leqslant'_\Theta x$. Then, we have $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ and $M_\Theta(\rho_\Theta + x\sigma_{\gamma_1}\mu) \subseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$. Moreover, $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ and $M_\Theta(\rho_\Theta + x\sigma_{\gamma_1}\mu) \subseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$ are compositions of elementary homomorphisms.

(2) Let $z, w \in W(\Theta)'$. If $M_\Theta(\rho_\Theta + z\mu) \subseteq M_\Theta(\rho_\Theta + w\mu)$, then $w \leqslant_\Theta z$.

We also have the following result.

Proposition 6.1.2. *Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is dominant integral and regular. Let $x, y \in {}^{e}W(\Theta)$. Then, we have $M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu) \not\subseteq M_\Theta(\rho_\Theta + x\mu)$ and $M_\Theta(\rho_\Theta + x\mu) \not\subseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$.*

Proof. Let $w_1 \in {}^{e}W(\Theta)$ be the longest element for $({}^{e}W(\Theta), {}^{e}S(\Theta))$. From proposition 4.4.1, we have $M_\Theta(\rho_\Theta + w_1\mu) \subseteq M_\Theta(\rho_\Theta + x\mu)$ and $M_\Theta(\rho_\Theta + w_1\sigma_{\gamma_1}\mu) \subseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$.

We assume that n is even. Then, w_1 is the longest element for $(W(\Theta)', S(\Theta))$. So, $M_\Theta(\rho_\Theta + w_1\mu)$ is irreducible from [37] Proposition 8.5. Since $\gamma_1 \notin {}^{e}\Sigma_\Theta$, we have $M_\Theta(\rho_\Theta + w_1\mu) \not\subseteq M_\Theta(\rho_\Theta + w_1\sigma_{\gamma_1}\mu)$ from Proposition 4.1.4 and $w_1\sigma_{\gamma_1} = \sigma_{\gamma_1}w_1$. We assume that $M_\Theta(\rho_\Theta + w_1\mu) \subseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$. The Bernstein degree of a scalar generalized Verma module is one, it contains only one irreducible constituent of the maximal Gelfand-Kirillov dimension. So, from Proposition 1.4.1 $M_\Theta(\rho_\Theta + w_1\sigma_{\gamma_1}\mu) \subseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$

implies that $M_\Theta(\rho_\Theta + w_1\mu) \subseteq M_\Theta(\rho_\Theta + w_1\sigma_{\gamma_1}\mu)$. So, we have a contradiction. Hence, $M_\Theta(\rho_\Theta + w_1\mu) \nsubseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$. Since $M_\Theta(\rho_\Theta + w_1\mu) \subseteq M_\Theta(\rho_\Theta + x\mu)$, we have $M_\Theta(\rho_\Theta + x\mu) \nsubseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$. Next, we assume that $M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu) \subseteq M_\Theta(\rho_\Theta + x\mu)$. $M_\Theta(\rho_\Theta + w_1\mu)$ is the unique irreducible constituent of $M_\Theta(\rho_\Theta + x\mu)$ of the maximal Gelfand-Kirillov dimension from Proposition 1.4.1 and $M_\Theta(\rho_\Theta + w_1\mu) \nsubseteq M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$. So, we have $M_\Theta(\rho_\Theta + w_1\mu) \subseteq M_\Theta(\rho_\Theta + x\mu)/M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)$. On the other hand, $\mathrm{Dim}(M_\Theta(\rho_\Theta + x\mu)/M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu)) < \mathrm{Dim}(M_\Theta(\rho_\Theta + w_1\mu))$ from Proposition 1.4.1. So, we have a contradiction. This means that $M_\Theta(\rho_\Theta + y\sigma_{\gamma_1}\mu) \nsubseteq M_\Theta(\rho_\Theta + x\mu)$.

If we n is odd,then $w_1\sigma_{\gamma_1}$ is the longest element for $(W(\Theta)', S(\Theta))$. The proof of the proposition in this case is more or less similar to that for the case that n is even. So, we omit proving the proposition in this case. □

6.2 B_2 case

Next, we consider the case of $n = 2$. We fix $\mu \in \mathfrak{a}_\Theta^*$ such that $\rho_\Theta + \mu$ is regular dominant integral. In this case, ${}^eW(\Theta) = \{e, \sigma_{\gamma_2}, \sigma_{\gamma'}, \sigma_{\gamma_2}\sigma_{\gamma'}\} \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Since $\sigma_{\gamma_2} \leqslant_\Theta \sigma_{\gamma'}$ and $\sigma_{\gamma_2} \nleqslant'_\Theta \sigma_{\gamma'}$ hold, Proposition 6.1.1 and Proposition 6.1.2 are insufficient to determine whether $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\mu) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\mu)$ or not. However, as a corollary of Proposition 6.1.1 and Proposition 6.1.2, we easily have the following result.

Corollary 6.2.1. *Let $x, y \in W(\Theta)'$ such that $x \neq y$ and $(x, y) \neq (\sigma_{\gamma'}, \sigma_{\gamma_2})$. Then, we have*

(1) $M_\Theta(\rho_\Theta+x\mu) \subseteq M_\Theta(\rho_\Theta+y\mu)$ if and only if (x, y) appears in the following list.

$(\sigma_{\gamma_2}, e), (\sigma_{\gamma'}, e), (\sigma_{\gamma_2}\sigma_{\gamma'}, e), (\sigma_{\gamma_2}\sigma_{\gamma'}, \sigma_{\gamma'}), (\sigma_{\gamma_2}\sigma_{\gamma'}, \sigma_{\gamma_2}), (\sigma_{\gamma_2}\sigma_{\gamma_1}, \sigma_{\gamma_1}),$
$(\sigma_{\gamma'}\sigma_{\gamma_1}, \sigma_{\gamma_1}), (\sigma_{\gamma_2}\sigma_{\gamma'}\sigma_{\gamma_1}, \sigma_{\gamma_1}), (\sigma_{\gamma_2}\sigma_{\gamma'}\sigma_{\gamma_1}, \sigma_{\gamma'}\sigma_{\gamma_1}), (\sigma_{\gamma_2}\sigma_{\gamma'}\sigma_{\gamma_1}, \sigma_{\gamma_2}\sigma_{\gamma_1}).$

(2) If $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$, then it is a composition of elementary homomorphisms.

6.3 $\mathrm{B}_{n,1,n-2}$

We fix notation as follows. Let $n \geqslant 3$. Let $\mathfrak{g}$ be a simple Lie algebra of the type B_n. We can choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{e_i \mid 1 \leqslant i \leqslant n\}.$$

If we put $\alpha_i = e_i - e_{i+1}$ $\quad (1 \leqslant i < n)$ and $\alpha_n = e_n$, then $\Pi = \{\alpha_1, ..., \alpha_n\}$. We put $\Theta = \Pi - \{\alpha_1, \alpha_2\}$.

$$\bigcirc - \bigcirc - \odot - \odot - \cdots - \odot \Rightarrow \odot$$

Then, we have $\mathfrak{a}_\Theta^* = \{se_1 + te_2 \mid s, t \in \mathbb{C}\}$ and $\rho_\Theta = \sum_{i=3}^m \frac{2m-2i+1}{2} e_i$. We put $\gamma_1 = \alpha_2|_{\mathfrak{a}_\Theta} = e_2$, $\gamma_2 = \alpha_1|_{\mathfrak{a}_\Theta} = e_1 - e_2$, and $\gamma' = \sigma_{\gamma_1}\gamma_2 = e_1 + e_2$. Then, these notations are compatible with those in 6.2 and 6.3.

we put $\nu_0 = \frac{3}{2}e_1 + \frac{1}{2}e_2$ Then, $\rho_\Theta + \nu_0$ is integral and $\langle \nu_0, \gamma \rangle \geqslant 0$ for all $\gamma \in \Sigma_\Theta^+$. We put $\mu_1 = \frac{1}{2}e_1 + \frac{1}{2}e_2$, $\mu_2 = \frac{1}{2}e_1 - \frac{1}{2}e_2$, $\mu_3 = -\frac{1}{2}e_1 + \frac{1}{2}e_2$, and $\mu_4 = -\frac{1}{2}e_1 - \frac{1}{2}e_2$.

First, we have the following results by a straightforward computation. (Note: $\rho_\Theta + \sigma_{\gamma_2}\nu_0 = \frac{1}{2}e_1 + \frac{3}{2}e_2 + \sum_{i=3}^m \frac{2m-2i+1}{2}e_i$, $\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0 = -\frac{1}{2}e_1 + \frac{3}{2}e_2 + \sum_{i=3}^m \frac{2m-2i+1}{2}e_i = \rho_\Theta + \sigma_{\gamma_2}\nu_0 - e_1$.)

Lemma 6.3.1.

$$\{\rho_\Theta + \sigma_{\gamma_2}\nu_0 \pm e_i \mid 1 \leqslant k \leqslant n\} \cap \mathsf{P}_\Theta^{++} \cap W\dot{(}\rho_\Theta + \sigma_{\gamma_2}\nu_0) = \{\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0\}.$$

Let V be a natural representation of $\mathfrak{g}$. Namely, V is an irreducible representation of V with a highest weight e_1. The, the set of the weights of V is $\{\pm e_i \mid 1 \leqslant k \leqslant n\} \cup \{0\}$. We also easily have the following result.

Lemma 6.3.2. *There exists a sequence of $\mathfrak{p}_\Theta$-submodules of V*

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{2n+1} = V$$

which satisfies the following conditions (a)-(e).

(a) V_i/V_{i-1} is a one-dimensional $\mathfrak{p}_\Theta$-module such that $\mathfrak{n}_\Theta$ acts on it trivially for each $1 \leqslant i \leqslant 2n+1$.

(b) We denote by λ_i the highest weight of V_i/V_{i-1} as an $\mathfrak{l}_\Theta$-module. Then $\lambda_i = e_i$ for $1 \leqslant i \leqslant n$, $\lambda_{n+1} = 0$, and $\lambda_i = -e_{2n+2-i}$ for $n+2 \leqslant i \leqslant 2n+1$.

Lemma 6.3.3. *(1) $M_\Theta(\rho_\Theta + \mu_3)$ is irreducible.*

(2) $M_\Theta(\rho_\Theta + \mu_4)$ is irreducible.

Proof. (1) can be proved by Janzten's irreducibility condition ([18] Satz 3). (2) follows from [37] Proposition 8.5. □

Lemma 6.3.4.

$$M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \not\subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0).$$

Proof. We assume that $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$ and deduce a contradiction.

From Lemma 2.4.6 and the exactness of a translation functor, we have

$$M_\Theta(\rho_\Theta + \mu_2) \subseteq M_\Theta(\rho_\Theta + \mu_1).$$

From Proposition 4.2.2, we have $M_\Theta(\rho_\Theta + \mu_3) \subseteq M_\Theta(\rho_\Theta + \mu_2)$. So, we have $M_\Theta(\rho_\Theta + \mu_3) \subseteq M_\Theta(\rho_\Theta + \mu_1)$. From Proposition 4.2.2, we have $M_\Theta(\rho_\Theta + \mu_4) \subseteq M_\Theta(\rho_\Theta + \mu_1)$. Hence $M_\Theta(\rho_\Theta + \mu_1)$ has at least two distinct irreducible submodules. It contradicts Proposition 1.4.1 (3). □

Lemma 6.3.5.

$$M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \not\subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0).$$

Proof. Assuming $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$, we deduce a contradiction. Put $X = P_{\rho_\Theta+\nu_0}(M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \otimes V)$ and $Y = P_{\rho_\Theta+\nu_0}(M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0) \otimes V)$. Hence, we have $X \subseteq Y$.

We remark that $\sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0 - \sigma_{\gamma_2}\nu_0 = -e_1$. So, from Lemma 6.3.1 and Lemma 6.3.2, we see that there is a submodule $Y_1 \subsetneq Y$ such that $Y_1 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$ and $Y/Y_1 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0)$. On the other hand, $\sigma_{\gamma'}\sigma_{\gamma_1}\nu_0 - \sigma_{\gamma'}\nu_0 = e_1$. and it is the highest weight of the natural representation V. So, there is an embedding

$$\iota : M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq X \subseteq Y.$$

Considering composition with the canonical projection

$$q : Y \to Y/Y_1 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0),$$

we obtain a homomorphism

$$q \circ \iota : M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \to M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0).$$

We assume that $q \circ \iota = 0$. Then, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq Y_1$. However, Lemma 6.3.4 implies that it is a zero map. So, we have $q \circ \iota \neq 0$. From Theorem 2.1.1 (2), we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0)$. From Proposition 2.4.6 and the exactness of the translation functors, we have $M_\Theta(\rho_\Theta + \mu_2) \subseteq M_\Theta(\rho_\Theta + \mu_3)$. However, in our proof of Lemma 6.3.4, we see $M_\Theta(\rho_\Theta + \mu_3) \subseteq M_\Theta(\rho_\Theta + \mu_2)$. So, we obtained a contradiction. □

Proposition 6.3.6. *Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is dominant integral and regular.*

$$M_\Theta(\rho_\Theta + \sigma_{\gamma'}\mu) \nsubseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\mu).$$

Proof. We assume that $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\mu) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\mu)$. From the translation principle, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\rho^\Theta) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\rho^\Theta)$. Since $\rho^\Theta - \nu_0$ is dominant integral, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$ from Lemma 2.4.7. It contradicts Lemma 6.3.5. □

Corollary 6.3.7. *Let $\lambda, \mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ and $\rho_\Theta + \lambda$ is integral and regular and $M_\Theta(\rho_\Theta + \mu) \subseteq M_\Theta(\rho_\Theta + \lambda)$. Then, $M_\Theta(\rho_\Theta + \mu) \subseteq M_\Theta(\rho_\Theta + \lambda)$ is a composition of some elementary homomorphisms.*

6.4 $\mathrm{E}_{7,27}$ and $\mathrm{E}_{8,18}$

We fix notation as follows.

First, we consider $\mathrm{E}_{7,27}$. Let $\mathfrak{g}$ be a simple Lie algebra of the type E_7.

We fix an orthonormal basis $e_1, ..., e_8$ in $\mathbb{R}^8$. We identify $\mathfrak{h}^*$ with $\{v \in \mathbb{R}^8 \mid \langle v, e_1 - e_2 \rangle = 0\}$ so that

$$\begin{aligned}&\Delta = \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j \mid 3 \leqslant i < j \leqslant 8\}\\ &\cup \left\{ \pm\frac{1}{2}\left(e_1 + e_2 + \sum_{i=3}^{8} \varepsilon_i e_i\right) \middle| \varepsilon_i = \pm 1 \text{ is for } 3 \leqslant i \leqslant 8 \text{ and } \textstyle\prod_{i=3}^8 \varepsilon_i = 1. \right\}\end{aligned}$$

We choose a positive system as follows.

$$\begin{aligned}&\Delta^+ = \{(e_1 + e_2)\} \cup \{e_i \pm e_j \mid 3 \leqslant i < j \leqslant 8\}\\ &\quad\cup \left\{ \frac{1}{2}\left(e_1 + e_2 + \sum_{i=3}^{8} \varepsilon_i e_i\right) \middle| \varepsilon_i = \pm 1 \text{ for } 3 \leqslant i \leqslant 8 \text{ and } \textstyle\prod_{i=3}^8 \varepsilon_i = 1. \right\}\end{aligned}$$

Put $\alpha_i = e_{i+2} - e_{i+3}$ for $1 \leqslant i \leqslant 5$, $\alpha_6 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, ..., \alpha_7\}$ is the set of simple roots in Δ^+.

$$\begin{array}{ccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 6 & - & 7 \\ & & & & & & | & & & & \\ & & & & & & 5 & & & & \end{array}$$

We consider the standard parabolic subalgebra of the type $\mathrm{E}_{7,27}$. Namely, we put $\Theta = \Pi - \{\alpha_2, \alpha_7\}$. Then, we have $\mathfrak{a}_\Theta^* = \{se_1 + se_2 + te_3 + te_4 \mid s, t \in \mathbb{C}\}$ and $\rho_\Theta = \frac{1}{2}e_3 - \frac{1}{2}e_4 + 3e_5 + 2e_6 + e_7$. We put $\gamma_1 = \alpha_2|_{\mathfrak{a}_\Theta} = \frac{1}{2}e_3 + \frac{1}{2}e_4$,

$\gamma_2 = \alpha_7|_{\mathfrak{a}_\Theta} = \frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4$, and $\gamma' = \sigma_{\gamma_1}\gamma_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4$. Then, these notations are compatible with those in 6.2 and 6.3. we put $\nu_0 = \frac{3}{2}e_1 + \frac{3}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4$ Then, $\rho_\Theta + \nu_0 = \frac{3}{2}e_1 + \frac{3}{2}e_2 - e_4 + 3e_5 + 2e_6 + e_7$ is integral and $\rho - (\rho_\Theta + \nu_0) = 7e_1 + 7e_2 + 5e_3 + 3e_4 + 3e_5 + 2e_6 + e_7$ is dominant integral. We put $\mu_1 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4$, $\mu_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4$, $\mu_3 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4$, $\mu_4 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4$.

Next, we consider the case of $\mathrm{E}_{8,18}$. Let $\mathfrak{g}$ be a simple Lie algebra of the type E_7. We fix an orthonormal basis $e_1, ..., e_8$ in $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 8\}$$
$$\cup \left\{ \pm\frac{1}{2}\left(\sum_{i=1}^{8} \varepsilon_i e_i\right) \middle| \varepsilon_i = \pm 1 \text{for } 1 \leqslant i \leqslant 8 \text{ and } \textstyle\prod_{i=1}^{8} \varepsilon_i = -1. \right\}$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 8\}$$
$$\cup \left\{ \frac{1}{2}\left(e_1 + \sum_{i=2}^{8} \varepsilon_i e_i\right) \middle| \varepsilon_i = \pm 1 \text{ is for } 2 \leqslant i \leqslant 8 \text{ and } \textstyle\prod_{i=2}^{8} \varepsilon_i = -1. \right\}$$

Put $\alpha_i = e_{i+1} - e_{i+2}$ for $1 \leqslant i \leqslant 6$, $\alpha_7 = e_7 + e_8$, and $\alpha_8 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, ..., \alpha_8\}$ is the set of simple roots in Δ^+.

$$\begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 7 & - & 8 \\ & & & & & & & & | & & & & \\ & & & & & & & & 6 & & & & \end{array}$$

We consider the standard parabolic subalgebra of the type $\mathrm{E}_{8,18}$. Namely, we put $\Theta = \Pi - \{\alpha_1, \alpha_8\}$. Then, we have $\mathfrak{a}_\Theta^* = \{se_1 + te_2 \mid s, t \in \mathbb{C}\}$ and $\rho_\Theta = 5e_3 + 4e_4 + 3e_5 + 2e_6 + e_7$. We put $\gamma_1 = \alpha_8|_{\mathfrak{a}_\Theta} = e_1 - e_2$, $\gamma_2 = \alpha_1|_{\mathfrak{a}_\Theta} = e_2$, and $\gamma' = \sigma_{\gamma_1}\gamma_2 = e_1$. Then, these notations are compatible with those in 6.2 and 6.3. we put $\nu_0 = 2e_1 + e_2$. It is dominant with respect to Σ_Θ^+. Then, $\rho_\Theta + \nu_0 = 2e_1 + e_2 + 5e_3 + 4e_4 + 3e_5 + 2e_6 + e_7$ is integral and $\rho - (\rho_\Theta + \nu_0) = 21e_1 + 5e_2$ is dominant integral. We put $\mu_1 = e_1$, $\mu_2 = -e_2$, $\mu_3 = e_2$, $\mu_4 = -e_1$.

Hereafter, we assume that $\mathfrak{g}$ is of either the type E_7 or E_8. We treat these case at the same time.

First, we have the following results by a straightforward computation.

Lemma 6.4.1.

$$\{\rho_\Theta + \sigma_{\gamma_2}\nu_0 + \beta \mid \beta \in \Delta \cup \{0\}\} \cap \mathsf{P}_\Theta^{++} \cap \dot{W}(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$$
$$= \{\rho_\Theta + \sigma_{\gamma_2}\nu_0, \rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0\}.$$

We also have the following result.

Lemma 6.4.2. *We regard $\mathfrak{g}$ as an adjoint representation.*

(1) There exists a sequence of $\mathfrak{p}_\Theta$-submodules of $\mathfrak{g}$

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = \mathfrak{g}$$

and a sequence of roots $\beta_1, ..., \beta_k \in \Delta$ which satisfies the following conditions (a)-(e).

(a) V_i/V_{i-1} is an irreducible $\mathfrak{p}_\Theta$-module such that $\mathfrak{n}_\Theta$ acts on it trivially for each $1 \leqslant i \leqslant k$.

(b) As a $\mathfrak{l}_\Theta$-module, V_i/V_{i-1} has a highest weight β_i for each $1 \leqslant i \leqslant k$.

(c) β_1 is the highest root $e_1 + e_2$.

(d) β_k is the lowest root $-e_1 - e_2$.

(e) There exist some $1 < h_1 < h_2 < k$ such that $\beta_{h_1} = \beta_{h_2} = 0$ and $\beta_i \neq 0$ for all $i \neq h_1.h_2$.

(2) V_1, V_{h_1}/V_{h_1-1}, V_{h_2}/V_{h_2-1}, V_k/V_{k-1} are all one-dimensional.

Proof. The existence of $V_1, ..., V_k$ satisfying (a)-(d) is proved by a standard argument. For $H \in \mathfrak{h}$, $[\mathfrak{l}_\Theta \cap \mathfrak{n}, H] = 0$ if and only if $H \in \mathfrak{a}_\Theta$. Since $\dim \mathfrak{a}_\Theta = 2$, $V_1, ..., V_k$ satisfies (e) also. From $\pm(e_1 + e_2), 0 \in \mathfrak{a}_\Theta^*$, we easily have (2). □

Lemma 6.4.3. *(1) $M_\Theta(\rho_\Theta + \mu_3)$ is irreducible.*

(2) $M_\Theta(\rho_\Theta + \mu_4)$ is irreducible.

Proof. (1) can be proved by Janzten's irreducibility condition ([18] Satz 3). (2) follows from [37] Proposition 8.5. □

Lemma 6.4.4.

$$M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \nsubseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0).$$

Proof. We assume that $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$ and deduce a contradiction.

From Lemma 2.4.6 and the exactness of a translation functor, we have

$$M_\Theta(\rho_\Theta + \mu_2) \subseteq M_\Theta(\rho_\Theta + \mu_1).$$

From Proposition 4.2.2, we have $M_\Theta(\rho_\Theta + \mu_3) \subseteq M_\Theta(\rho_\Theta + \mu_2)$. So, we have $M_\Theta(\rho_\Theta + \mu_3) \subseteq M_\Theta(\rho_\Theta + \mu_1)$. From Proposition 4.2.2, we have $M_\Theta(\rho_\Theta + \mu_4) \subseteq M_\Theta(\rho_\Theta + \mu_1)$. Hence $M_\Theta(\rho_\Theta + \mu_1)$ has at least two distinct irreducible submodules. It contradicts Proposition 1.4.1 (3). □

Lemma 6.4.5.

$$M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \not\subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0).$$

Proof. Assuming $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$, we deduce a contradiction. Put $X = P_{\rho_\Theta+\nu_0}(M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \otimes \mathfrak{g})$ and $Y = P_{\rho_\Theta+\nu_0}(M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0) \otimes \mathfrak{g})$. Hence, we have $X \subseteq Y$.

We remark that $\sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0 - \sigma_{\gamma_2}\nu_0 = -e_1 - e_2$. So, from Lemma 6.4.1 and Lemma 6.4.2, we see that there is a sequence of submodules $Y_1 \subsetneq Y_2 \subsetneq Y$ such that $Y_1 \cong Y_2/Y_1 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$ and $Y/Y_2 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0)$. On the other hand, $\sigma_{\gamma'}\sigma_{\gamma_1}\nu_0 - \sigma_{\gamma'}\nu_0 = e_1 + e_2$. and it is the highest weight of the adjoint representation $\mathfrak{g}$. So, there is an embedding

$$\iota : M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq X \subseteq Y.$$

Considering composition with the canonical projection

$$q : Y \to Y/Y_2 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0),$$

we obtain a homomorphism

$$q \circ \iota : M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \to M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0).$$

We assume that $q \circ \iota = 0$. Then, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq Y_2$. Again we consider the composition $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq Y_2 \to Y_2/Y_1 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$. However, Lemma 6.4.4 implies that it is a zero map. So, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq Y_1 \cong M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$. It contradicts Lemma 6.3.4, So, we have $q \circ \iota \neq 0$. From Theorem 2.1.1 (2), we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\sigma_{\gamma_1}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\sigma_{\gamma_1}\nu_0)$. From Proposition 2.4.6 and the exactness of the translation functors, we have $M_\Theta(\rho_\Theta + \mu_2) \subseteq M_\Theta(\rho_\Theta + \mu_3)$. However, in our proof of Lemma 6.4.4, we see $M_\Theta(\rho_\Theta + \mu_3) \subseteq M_\Theta(\rho_\Theta + \mu_2)$. So, we obtained a contradiction. □

Proposition 6.4.6. *Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is dominant integral and regular.*

$$M_\Theta(\rho_\Theta + \sigma_{\gamma'}\mu) \not\subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\mu).$$

Proof. We assume that $M_\Theta(\rho_\Theta+\sigma_{\gamma'}\mu) \subseteq M_\Theta(\rho_\Theta+\sigma_{\gamma_2}\mu)$. From the translation principle, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\rho^\Theta) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\rho^\Theta)$. Since $\rho^\Theta - \nu_0$ is dominant integral, we have $M_\Theta(\rho_\Theta + \sigma_{\gamma'}\nu_0) \subseteq M_\Theta(\rho_\Theta + \sigma_{\gamma_2}\nu_0)$ from Lemma 2.4.7. It contradicts Lemma 6.4.5. □

Corollary 6.4.7. *Let $\lambda, \mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta+\mu$ and $\rho_\Theta+\lambda$ is integral and regular and $M_\Theta(\rho_\Theta + \mu) \subseteq M_\Theta(\rho_\Theta + \lambda)$. Then, $M_\Theta(\rho_\Theta + \mu) \subseteq M_\Theta(\rho_\Theta + \lambda)$ is a composition of some elementary homomorphisms.*

6.5 $\mathrm{F}_{4,14}$

We consider the root system Δ for a simple Lie algebra $\mathfrak{g}$ of the type F_4. (For example, see [21] p691.) We can choose an orthonormal basis $e_1, ..., e_4$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 4\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant 4\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 4\} \cup \{e_i \mid 1 \leqslant i \leqslant 4\} \cup \left\{ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

Put $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_2 - e_3$. Then, $\Pi = \{\alpha_1, ..., \alpha_4\}$.

$$1 \; - \; 2 \; \Leftarrow \; 3 \; - \; 4$$

We consider the standard parabolic subalgebra of the type $\mathrm{F}_{4,14}$. Namely, we put $\Theta = \Pi - \{\alpha_1, \alpha_4\}$. Then, we have $\mathfrak{a}_\Theta^* = \{se_1 + te_2 \mid s, t \in \mathbb{C}\}$ and $\rho_\Theta = \frac{3}{2}e_3 - \frac{1}{2}e_4$. We put $\gamma_1 = \alpha_1|_{\mathfrak{a}_\Theta} = \frac{1}{2}e_1 - \frac{1}{2}e_2$, $\gamma_2 = \alpha_4|_{\mathfrak{a}_\Theta} = e_2$. In this setting, ${}^{ur}\Sigma_\Theta$ is of the type B_2. On ${}^{e}\Sigma_\Theta = \emptyset$ and ${}^{e}W(\Theta) = \{e\}$. We put $\mu_1 = \frac{1}{2}e_1 + \frac{1}{2}e_2$, $\mu_2 = \frac{1}{2}e_1 - \frac{1}{2}e_2$, $\mu_3 = -\frac{1}{2}e_1 + \frac{1}{2}e_2$, $\mu_4 = -\frac{1}{2}e_1 - \frac{1}{2}e_2$. From Janzten's irreducibility criterion, we can show the following result.

Lemma 6.5.1. *$M_\Theta(\rho_\Theta + \mu_i)$ is irreducible for each $1 \leqslant i \leqslant 4$.*

Finally, we prove the following result.

Proposition 6.5.2. *Let $\lambda, \nu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \lambda$ and $\rho_\Theta + \nu$ are regular integral and $\lambda \neq \nu$. Then $M_\Theta(\rho_\Theta + \nu) \nsubseteq M_\Theta(\rho_\Theta + \lambda)$.*

Proof. We assume that $M_\Theta(\rho_\Theta + \nu) \subseteq M_\Theta(\rho_\Theta + \lambda)$. From Proposition 2.3.2, there exists some $x \in W(\Theta)$ such that $\nu = x\lambda$. We easy to see there is unique $1 \leqslant i \leqslant 4$ (resp. $1 \leqslant i \leqslant 4$) such that λ and μ_i (resp. $x\lambda$ and μ_j) satisfy the condition (T) in 2.4. From proposition 2.4.7, we have $M_\Theta(\rho_\Theta + \mu_j) \subseteq M_\Theta(\rho_\Theta + \mu_i)$. From Lemma 6.5.1, we have $i = j$. Then, we may apply Proposition 4.1.4 and $M_\Theta(\rho_\Theta + \nu) \subseteq M_\Theta(\rho_\Theta + \lambda)$is an elementary homomorphism. However, it contradicts ${}^{e}\Sigma_\Theta = \emptyset$. $\square$

§ 7. Type A case

7.1 Some notations

In this section, we assume that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of the diagonal matrices and let $\mathfrak{b}$ be the Borel subalgebra of

$\mathfrak{g}$ consisting of the upper triangular matrices. We choose Δ^+ corresponding to $\mathfrak{b}$. Then we can choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ such that

$$\Delta^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n, i \neq j\}.$$

If we put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i < n)$, then $\Pi = \{\alpha_1, ..., \alpha_{n-1}\}$. We identify the Weyl group W with the n-th symmetric group $\mathfrak{S}_n$ via $\sigma e_i = e_{\sigma(i)}$ $(1 \leqslant i \leqslant n)$.

7.2 Almost normal parabolic subalgebras

We fix $\Theta \subsetneq \Pi$. Put $\Theta_0 = \{\beta \in \Theta \mid w\beta = \beta \ (w \in W(\Theta))\}$ and put $\Theta_1 = \Theta - \Theta_0$. We put $\Phi_{\Theta_0} = \Delta \cap \mathfrak{a}^*_{\Theta_0}$ and $\Phi^+_{\Theta_0} = \Delta^+ \cap \mathfrak{a}^*_{\Theta_0}$. Then, Φ_{Θ_0} is a sub root system of Δ and $\Phi^+_{\Theta_0}$ is a positive system of Φ_{Θ_0}. We denote by $\Pi[\Theta_0]$ the basis of $\Phi^+_{\Theta_0}$. We easy to see that $\Theta_1 \subseteq \Pi[\Theta_0]$. We call Θ *almost normal* (resp. *almost seminormal*) if Θ_1 is normal (resp. seminormal) as a subset of $\Pi[\Theta_0]$.

Let $1 \leqslant s_1 < s_2 < \cdots < s_{k-1} < n$ be such that $\Pi - \Theta = \{\alpha_{s_1}, ..., \alpha_{s_{k-1}}\}$. We put $s_0 = 0$ and $s_k = n$. We also put $n_i = s_i - s_{i-1}$ for $1 \leqslant i \leqslant k$. Then, we easily see

$$\mathfrak{l}_\Theta \cong \mathfrak{gl}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_k, \mathbb{C}). \quad (*)$$

For a positive integer q, we put $m_q = \mathrm{card}\{i \mid 1 \leqslant i \leqslant k, n_i = q\}$. We enumerate $\{q \mid m_q \neq 0\} = \{q_1, ..., q_u\}$ so that $q = 1 < \cdots < q_u$. Since an element of $W(\Theta)$ induces a permutation of the direct summand of $(*)$, we have $W(\Theta) \cong \mathfrak{S}_{m_{q_1}} \times \cdots \times \mathfrak{S}_{m_{q_u}}$. From Proposition 3.3.2 (1), we easily see :

Proposition 7.2.1. *Θ is almost normal if and only if there is some positive integer p such that $m_q \leqslant 1$ for all $q \neq p$. In this case, we have $W(\Theta) \cong \mathfrak{S}_{m_p}$.*

Let $1 \leqslant s'_1 < s'_2 < \cdots < s'_{m-1} < n$ be such that $\Pi - \Theta_0 = \{\alpha_{s'_1}, ..., \alpha_{s'_{m-1}}\}$.
We define $\beta_1, ..., \beta_{m-1} \in \Delta^+$ as follows.

$$\beta_i = \begin{cases} \alpha_{s'_i} & \text{if } s'_i + 1 = s'_{i+1} \text{ or } s'_i = n-1 \\ \sum_{j=s'_i}^{s'_{i+1}} \alpha_j & \text{otherwise} \end{cases}$$

Then, we have $\Pi[\Theta_0] = \{\beta_1, ..., \beta_{m-1}\}$. Φ_{Θ_0} is clearly a root system of the type A_{m-1}. We put $S[\Theta_0] = \{s_{\beta_1}, ..., s_{\beta_{m-1}}\}$ and denote by $W[\Theta_0]$ the subgroup of W generated by $S[\Theta_0]$. $W[\Theta_0]$ is the Weyl group for the root system Φ_{Θ_0}. We denote by $\leqslant_{[\Theta_0]}$ the Bruhat ordering for the Coxeter system $(W[\Theta_0], S[\Theta_0])$. Since $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, we easily see $W(\Theta) \subseteq W[\Theta_0]$. From Theorem 3.4.5, we have:

Lemma 7.2.2. *We assume that Θ is almost seminormal. Then, for $x, y \in W(\Theta)$, $x \leqslant_{[\Theta_0]} y$ if and only if $x \leqslant_\Theta y$.*

7.3 Comparison of Bruhat orderings

First, we recall a famous description of the Bruhat ordering of a type A Weyl group. For a positive integer n, We put $[n] = \{1, ..., n\}$. Let S be a nonempty subset of $[n]$ and let $\mathbb{R}^S$ be the set of the functions of S to $\mathbb{R}$. We denote by $\mathfrak{S}(S)$ the group consisting of the bijection of S to S. Then, $\mathfrak{S}(S)$ acts on $\mathbb{R}^S$ as follows.

$$\tau f(s) = f(\tau^{-1}(s)) \quad (f \in \mathbb{R}^S, \tau \in \mathfrak{S}(S)).$$

We enumerate the elements of S as follows.

$$S = \{\ell_1, ...\ell_h\} \ \ \ell_1 < \cdots < \ell_h.$$

We put $S_r = \{\ell_1, ..., \ell_r\}$ for $1 \leqslant r \leqslant h$. $\mathfrak{S}(S)$ is regarded as a Coxeter group with the set of generators consisting of the transposition of ℓ_i and ℓ_{i+1} for $1 \leqslant i < r$. We denote by $\leqslant_S$ the Bruhat ordering of $\mathfrak{S}(S)$.

For $f \in \mathbb{R}^S$, we choose $\tau \in \mathfrak{S}(S)$ such that τf satisfies $\tau f(s) \geqslant \tau f(t)$ for all $s, t \in S$ such that $s \leqslant t$. Since τf depends only on f, we write f^* for τf. Let $f_1, f_2 \in \mathbb{R}^S$. We write $f_1 \preceq f_2$ if $f_1^*(s) \leqslant f_2^*(s)$ for all $s \in S$. We write $f_1 \trianglelefteq f_2$ if $f_1|_{S_r} \preceq f_2|_{S_r}$ for all $1 \leqslant r \leqslant h$.

The following lemma is easy.

Lemma 7.3.1. *Let S be a nonempty subset of $[n]$ and let $f_1, f_2 \in \mathbb{R}^{[n]}$ be such that $f_1|_{[n]-S} = f_2|_{[n]-S}$. Then, $f_1 \trianglelefteq f_2$ if and only if $f_1|_S \trianglelefteq f_2|_S$.*

The following characterization of the Bruhat ordering is well-known (for example, see [13]).

Proposition 7.3.2. *We fix a strictly decreasing function $f_0 \in \mathbb{R}^S$. For $x, y \in \mathfrak{S}(S)$, we have $x \leqslant_S y$ if and only if $yf_0 \trianglelefteq xf_0$*

We prove the following result.

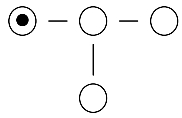
Proposition 7.3.3. *Let $w \in W$ be such that $w^{-1}\beta \in \Delta^+$ for all $\beta \in \Phi^+_{\Theta_0}$. For any $x, y \in W[\Theta_0]$, $xw \leqslant yw$ if and only if $x \leqslant_{[\Theta_0]} y$.*

Proof. We put $S^c = \{i \in [n] \mid \ \alpha_i \in \Theta_0 \text{ or } \alpha_{i-1} \in \Theta_0\}$ and $S = [n] - S^c$. If we identify e_i with i for $1 \leqslant i \leqslant n$, we have an identification of W with $\mathfrak{S}_n$. Then, $W[\Theta_0]$ is identified with $\mathfrak{S}(S)$. We fix a strictly monotone decreasing function $f_0 \in \mathbb{R}^{[n]}$ and $x, y \in W[T_0]$. Hence $xw \leqslant yw$ if and only if $ywf_0 \trianglelefteq xwf_0$. Since $x, y \in \mathfrak{S}(S)$, $ywf_0|_{S^c} = xwf_0|_{S^c}$. So, we have that,from 7.3.1, $xw \leqslant yw$ if and only if $ywf_0|_S \trianglelefteq xwf_0|_S$. Since $wf_0|_S$ is also strictly monotone decreasing, using Proposition 7.3.2, we have the proposition. □

From Lemma 7.2.2 and Proposition 7.3.3, we immediately have:

Corollary 7.3.4. *Let $\Theta \subsetneq \Pi$ be almost seminormal and let $w \in W$ be such that $w^{-1}\beta \in \Delta^+$ for all $\beta \in \Phi^+_{\Theta_0}$. Then, for $x, y \in W(\Theta)$, $xw \leqslant yw$ if and only if $x \leqslant_\Theta y$.*

Remark The corresponding statement to Proposition 7.3.4 is not necessarily correct for a general reductive Lie algebra $\mathfrak{g}$. A counterexample is as follows.



For the type A Weyl group, each involution is a Duflo involution. So, Θ-useful root is always Θ-excellent. So, we obtain he following result from Corollary 7.3.4 in a similar way to the proof of Theorem 5.1.3.

Theorem 7.3.5. *Let $\Theta \subsetneq \Pi$ be almost normal.*

(1) Let $\mu \in \mathfrak{a}^*_\Theta$ be such that $\rho_\Theta + \mu$ is regular integral and $\langle \mu, \gamma \rangle > 0$ for all ${}^{ru}\Sigma^+_\Theta$. Then, for $x, y \in W(\Theta)$ we have $x \leqslant_\Theta y$ if and only if $M_\Theta(\rho_\Theta + y\mu) \subseteq M_\Theta(\rho_\Theta + x\mu)$.

(2) Any nonzero homomorphism between scaler generalized Verma modules for Θ with a regular integral infinitesimal character is a composition of elementary homomorphisms.

We consider some special cases.

Corollary 7.3.6. *Let $p+q = n$ and let $\Theta \subseteq$ be such that $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of a real form $\mathfrak{u}(p,q)$ of $\mathfrak{gl}(n.\mathbb{C})$. Then, nonzero homomorphism between scaler generalized Verma modules for Θ with a regular integral infinitesimal character is a composition of elementary homomorphisms.*

Let n be a positive integer such that $2 \leqslant n \leqslant 5$. Then, we easily see that any parabolic subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ is almost normal.

Corollary 7.3.7. *Let $2 \leqslant n \leqslant 5$ and let $\mathfrak{g} = \mathfrak{gl}(n.\mathbb{C})$. Then, nonzero homomorphism between scaler generalized Verma modules with a regular integral infinitesimal character is a composition of elementary homomorphisms.*

7.4 An example in $\mathfrak{gl}(6, \mathbb{C})$

Let $\mathfrak{g} = \mathfrak{gl}(6, \mathbb{C})$.

Then we can choose an orthonormal basis $e_1, ..., e_6$ of $\mathfrak{h}^*$ as in 7.1. So, $\Pi = \{\alpha_1, ..., \alpha_5\}$, where $\alpha_i = e_1 - e_{i+1}$. We write $(abcdfg)$ for $ae_1 + be_2 +$

$ce_3+de_4+fe_5+ge_6$. We put $\bar{\rho}=(654321)=6e_1+5e_2+4e_3+3e_4+2e_5+e_6$. Put $\Theta=\{\alpha_1,\alpha_5\}$.

$$\odot - \bigcirc - \bigcirc - \bigcirc - \odot$$

Then $\mathfrak{l}_\Theta \cong \mathfrak{gl}(2,\mathbb{C})\oplus\mathfrak{gl}(1,\mathbb{C})\oplus\mathfrak{gl}(1,\mathbb{C})\oplus\mathfrak{gl}(2,\mathbb{C})$ and Θ is not almost semi-normal. For this Θ, two Bruhat orderings $\leqslant$ and $\leqslant_\Theta$ are not compatible. A counterexample is given as follows. Let $x\in W$ and $y\in W$ be such that $x\bar{\rho}=(653421)$ and $y\bar{\rho}=(214365)$. Then $x,y\in W(\Theta)$, $x\leqslant y$, and $x\not\leqslant_\Theta y$. The following result means that this example does not produce a counterexample to Conjecture 4.1.2.

Proposition 7.4.1. *$M_\Theta(y\bar{\rho})\not\subseteq M_\Theta(x\bar{\rho})$.*

Proof. We assume that $M_\Theta(y\bar{\rho})\subseteq M_\Theta(x\bar{\rho})$. Namely, $M_\Theta((214365))\subseteq M_\Theta((653421))$. Let V be a natural representation of $\mathfrak{g}$ and V^* its contragradient. Then the set of weights of $\wedge^2V^*$ is $\{-e_i-e_j \mid 1\leqslant i<j\leqslant 6\}$. We consider a translation functor $T_{(653421)}^{(543421)}(M)=P_{(543421)}(M\otimes\wedge^2V^*)$. We easily see that $W\cdot(543421)\cap\{(653421)-e_i-e_j \mid 1\leqslant i<j\leqslant 6\}=\{(543421)\}$ and $W\cdot(214354)\cap\{(214365)-e_i-e_j \mid 1\leqslant i<j\leqslant 6\}=\{(214354)\}$. Hence, we have $T_{(653421)}^{(543421)}(M_\Theta(653421))=M_\Theta((543421))$ and $T_{(653421)}^{(543421)}(M_\Theta(214365))=M_\Theta((214354))$. The exactness of the translation functors implies :

$$M_\Theta((214354))\subseteq M_\Theta((543421)).$$

Applying $T_{(543421)}^{(434321)}$, we have

$$M_\Theta((214343))\subseteq M_\Theta((433421))$$

in a similar way. Next we apply $T_{(434321)}^{(434332)}$ and $T_{(434332)}^{(434343)}$ successively, we finally have:

$$M_\Theta((434343))\subseteq M_\Theta((433443))$$

. However, it is impossible since $(433443)-(434343)=e_3+e_4$ is not a sum of negative roots. $\square$

§ 8. Class one setting

8.1 Background

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$. We choose $\mathfrak{h}$ as a complexification of a maximally split Cartan subalgebra of $\mathfrak{g}_0$. We choose $\mathfrak{b}$ and $\Theta\subseteq\Pi$ such that $\mathfrak{p}_\Theta$ is the complexification of a minimal parabolic subalgebra of $\mathfrak{g}_0$. If $\mathfrak{a}_\Theta$ is Iwasawa's $\mathfrak{a}$ (namely real split torus with

respect to $\mathfrak{g}_0$), we can easily see that Θ is normal from the classification. In this case, $W(\Theta)$ coincides with the little Weyl group of the restricted root system. However, if $\mathfrak{g}_0$ is not quasi-split and $\mathfrak{a}_\Theta$ is not a real split torus, then Θ is not normal. If $\mathfrak{g}_0 = \mathfrak{su}(p,q)$, we have Corollary 7.3.6. So, we consider the remaining two cases $\mathfrak{so}^*(4n+2)$ and $\mathfrak{e}_{6(-14)}$.

8.2 General setting

At first, we consider rather general situation. Let $\mathfrak{g}$ be a complex simple Lie algebra and let τ be an outer automorphism preserving $\mathfrak{h}$ and $\mathfrak{b}$. Such an automorphism comes from a symmetry of the Dynkin diagram corresponding to Π. We assume the order of τ is two. We also denote by τ the induced automorphism of Δ, W, and $\mathfrak{h}^*$. We put ${}^\tau\mathfrak{h} = \{X \in \mathfrak{h} \mid \tau(X) = X\}$, ${}^\tau W = \{w \in W \mid \tau(w) = w\}$, ${}^\tau\Delta = \{\alpha|_{{}^\tau\mathfrak{h}} \mid \alpha \in \Delta\}$, and ${}^\tau\Delta^+ = \{\alpha|_{{}^\tau\mathfrak{h}} \mid \alpha \in \Delta^+\}$. For $\alpha \in \Delta$, we denote by ξ_α the longest element of a parabolic subgroup $W_{\{\alpha,\tau(\alpha)\}}$. Namely, we have

$$\xi_\alpha = \begin{cases} s_\alpha & \text{if } \alpha = \tau(\alpha), \\ s_\alpha s_{\tau(\alpha)} & \text{if } \langle \alpha, \tau(\alpha)\rangle = 0, \\ s_\alpha s_{\tau(\alpha)} s_\alpha & \text{if } \langle \alpha, \tau(\alpha)^\vee\rangle = -1. \end{cases}$$

Put ${}^\tau S = \{xi_\alpha \mid \alpha \in \Pi\}$ and ${}^\tau\Pi = \{\alpha|_{{}^\tau\mathfrak{h}} \mid \alpha \in \Pi\}$. The following result is known.

Proposition 8.2.1. ([34], [32])

(1) For $\alpha \in \Delta$, $\xi_\alpha|_{{}^\tau\mathfrak{h}}$ is the reflection with respect to $\alpha|_{{}^\tau\mathfrak{h}}$.

(2) $({}^\tau W, {}^\tau S)$ is a Coxeter system.

(3) ${}^\tau W$ can be regarded as a reflection group for a root system ${}^\tau\Delta$.

We denote by $\leqslant_\tau$ the Bruhat ordering for $({}^\tau W, {}^\tau S)$. As before, we denote by $\leqslant$ for the Bruhat ordering for W. We quote:

Theorem 8.2.2. ([32]) *Let $x, y \in {}^\tau W$. Then $x \leqslant y$ if and only if $x \leqslant_\tau y$.*

We fix $\Theta \subseteq \pi$ such that $\tau(\Theta) = \Theta$. We denote by ${}^\tau\Theta = \{\alpha|_{{}^\tau\mathfrak{h}} \mid \alpha \in \Theta\}$. We put ${}^\tau\mathfrak{a}_\Theta = \mathfrak{a}_\Theta \cap {}^\tau\mathfrak{h}$ and ${}^\tau W({}^\tau\Theta) = \{w \in {}^\tau W \mid w{}^\tau\Theta = {}^\tau\Theta\}$. We put ${}^\tau\Sigma_\Theta^+ = \{\beta|_{{}^\tau\mathfrak{a}_\Theta} \mid \beta \in {}^\tau\Delta^+\} - \{0\}$.

Applying Theorem 3.4.5 and Theorem 8.2.2, we obtain the following result in a similar way to Theorem 5.1.3.

Corollary 8.2.3. *We assume that the following conditions (a)-(c).*

(a) $^{\tau}\Theta$ *is a normal subset of* $^{\tau}\Pi$.

(b) As a subgroup of W, $W(\Theta)$ *coincides with* $^{\tau}W(^{\tau}\Theta)$.

(c) $W(\Theta) = {}^{e}W(\Theta)$. *(In particular,* $W(\Theta) = W(\Theta)'$ *holds.)*

We fix some $\mu \in {}^{\tau}\mathfrak{a}^{*}_{\Theta}$ *such that* $\rho_\Theta + \mu$ *is regular integral and* $\langle \nu, \gamma \rangle > 0$ *for all* $\gamma \in {}^{\tau}\Sigma^{+}_{\Theta}$. *Then, for all* $x, y \in W(\Theta)$, $x \leqslant_\Theta y$ *if and only if* $x \leqslant_\Theta y$.

8.3 $\mathfrak{so}^{*}(4m+2)$

Let $\mathfrak{g}$ be a complex simple Lie algebra of the type $D_{2m+1} \quad (n \geqslant 2)$. Then we can choose an orthonormal basis $e_1, ..., e_{2m+1}$ of $\mathfrak{h}^{*}$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 2m+1\}.$$

We choose a positive system as follows.

$$\Delta^{+} = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 2m+1\}.$$

If we put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i \leqslant 2m)$ and $\alpha_{2m+1} = e_{2m} + e_{2m+1}$, then $\Pi = \{\alpha_1, ..., \alpha_{2m+1}\}$.

Let $\Theta \subseteq \Pi$ be such that $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of $\mathfrak{so}^{*}(4m+2)$. Namely, $\Theta = \{\alpha_{2i-1} \mid 1 \leqslant i \leqslant m\}$.

$$\begin{array}{ccccccccccccc} \odot & - & \bigcirc & - & \odot & - \cdots - & \odot & - & \bigcirc & - & \odot & - & \bigcirc \\ &&&&&&&&&& | && \\ &&&&&&&&&& \bigcirc && \end{array}$$

We choose τ so that it induces an automorphism of Δ described as follows.

$$\tau(e_i) = \begin{cases} e_i & (1 \leqslant i \leqslant 2m) \\ -e_i & (i = 2m+1). \end{cases}$$

We identify $^{\tau}\mathfrak{h}^{*}$ with $\{\sum_{i=1}^{2m} a_i e_i \mid a_1, ..., a_{2m} \in \mathbb{C}\} \subseteq \mathfrak{h}^{*}$. Hence, we have $^{\tau}\Theta \subseteq {}^{\tau}\Pi$ is of the type $\mathrm{B}_{2m,2,0}$ (cf. 3.3). In particular, the condition (a) in Corollary 8.2.3 holds in this case. We easily see that $^{\tau}\mathfrak{a}^{*}_{\Theta} = \{\sum_{i=1}^{m} a_i(e_{2i-1} + e_{2i}) \mid a_i \in \mathbb{C} \ (1 \leqslant i \leqslant m)\}$. We also see that $^{\tau}W(^{\tau}\Theta)$ is a Weyl group of the type B_m generated by $\{s_{e_{2i-1}-e_{2i+1}} s_{e_{2i}-e_{2i+2}} \mid 1 \leqslant i \leqslant m-1\} \cup \{s_{e_{2m-1}+e_{2m}}\}$. On the other hand, we see that $^{ru}\Delta^{+}_{\Theta} = {}^{e}\Delta^{+}_{\Theta} = \{e_{2i} - e_{2i+1} \mid 1 \leqslant i \leqslant m-1\} \cup \{e_{2i-1} + e_{2i} \mid 1 \leqslant i \leqslant m\}$. We also see :

$$\sigma_{e_{2i}-e_{2i+1}} = s_{e_{2i-1}-e_{2i+1}} s_{e_{2i}-e_{2i+2}} \quad (1 \leqslant i \leqslant m-1),$$

$$\sigma_{e_{2i-1}+e_{2i}} = s_{e_{2i-1}+e_{2i}} \quad (1 \leqslant i \leqslant m).$$

So, we have (b) and (c) in Corollary 8.2.3. Hence, we can apply Corollary 8.2.3 in this case. Moreover, $W(\Theta)$ can be identified with the little Weyl group for the restricted root system of the real form $\mathfrak{g}_0 = \mathfrak{so}^{*}(4m+2)$.

8.4 $\mathfrak{e}_{6,(-14)}$

We consider the root system Δ for a simple Lie algebra $\mathfrak{g}$ of the type E_6. Put $\kappa = \frac{1}{2\sqrt{3}}$. We can choose an orthonormal basis $e_1, ..., e_6$ of $\mathfrak{h}^*$ such that

$$\begin{aligned}\Delta = &\{e_i - e_j \mid 1 \leqslant i, j \leqslant 6, i \neq j\}\\ \cup &\left\{ \pm \sum_{i=1}^{6} \left(\kappa + \varepsilon_i \frac{1}{2}\right) e_i \,\middle|\, \varepsilon_i = \pm 1 \text{ for } 1 \leqslant i \leqslant 6, \operatorname{card}\{i \mid \varepsilon = 1, 1 \leqslant i \leqslant 6\} = 3 \right\}\\ &\cup \left\{ \pm 2\kappa \sum_{i=1}^{6} e_i \right\}.\end{aligned}$$

We choose a positive system as follows.

$$\begin{aligned}\Delta^+ = &\{e_i - e_j \mid 1 \leqslant i < j \leqslant 6\}\\ \cup &\left\{ \sum_{i=1}^{6} \left(\kappa + \varepsilon_i \frac{1}{2}\right) e_i \,\middle|\, \varepsilon_i = \pm 1 \text{ for } 1 \leqslant i \leqslant 6, \operatorname{card}\{i \mid \varepsilon = 1, 1 \leqslant i \leqslant 6\} = 3 \right\}\\ &\cup \left\{ 2\kappa \sum_{i=1}^{6} e_i \right\}.\end{aligned}$$

Put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i \leqslant 5)$ and $\alpha_6 = \sum_{i=1}^{3} \left(\kappa - \frac{1}{2}\right) e_i + \sum_{i=4}^{6} \left(\kappa + \frac{1}{2}\right) e_i$. Then, $\Pi = \{\alpha_1, ..., \alpha_6\}$. We put $\beta = 2\kappa \sum_{i=1}^{6} e_i$.

$$\begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 \\ & & & & | & & & & \\ & & & & 6 & & & & \end{array}$$

Let $\Theta \subseteq \Pi$ be such that $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of $\mathfrak{e}_{6,(-14)}$. Namely, $\Theta = \{\alpha_2, \alpha_3, \alpha_4\}$.

$$\begin{array}{ccccccccc} \bigcirc & - & \odot & - & \odot & - & \odot & - & \bigcirc \\ & & & & | & & & & \\ & & & & \bigcirc & & & & \end{array}$$

We choose τ so that it induces an automorphism of Δ described as follows.

$$\tau\left(\sum_{i=1}^{6} a_i e_i\right) = \sum_{i=1}^{6} \left(\frac{1}{3} \left(\sum_{j=1}^{6} a_j \right) - a_{7-i} \right) e_i.$$

Then, we have $\tau(\alpha_i) = \alpha_{6-i}$ for $1 \leqslant i \leqslant 5$ and $\tau(\alpha_6) = \alpha_6$. We identify ${}^{\tau}\mathfrak{h}^*$ with $\{\sum_{i=1}^{3}((a_4 + a_i)e_i + (a_4 - a_i)e_{7-i}) \mid a_1, ..., a_4 \in \mathbb{C}\} \subseteq \mathfrak{h}^*$. In fact, ${}^{\tau}\mathfrak{h}$ is

the complexification of " Iwasawa's $\mathfrak{a}$" (i.e. the real split part of the center of a Levi part of a minimal parabolic sualgebra) for a real form $\mathfrak{e}_{6(2)}$ of $\mathfrak{g}$. Hence, ${}^{\tau}\Delta$ is restricted root system for $\mathfrak{e}_{6(2)}$ and it is of the type F_4. We see that ${}^{\tau}\Theta \subseteq {}^{\tau}\Pi$ is of the type $\mathrm{F}_{4,14}$ (cf. 3.3). In particular, the condition (a) in Corollary 8.2.3 holds in this case.

We easily see that ${}^{\tau}\mathfrak{a}^*_{\Theta} = \{a(e_1 - e_6) + b\beta) \mid a, b \in \mathbb{C}\}$. We also see that ${}^{\tau}W({}^{\tau}\Theta)$ is a Weyl group of the type B_2. On the other hand, we see that ${}^{ru}\Delta^+_{\Theta} = {}^{e}\Delta^+_{\Theta} = \{e_1 - e_6, \beta, \alpha_6, s_{e=1-e_6}\alpha_6\}$ and ${}^{ru}\Sigma_{\Theta}$ is a root system of the type B_2. Hence $W(\Theta)' = {}^{e}W(\Theta)$ is the Weyl group of the type B_2. So, we have (c) in Corollary 8.2.3. From [15], we see $W(\Theta) = W(\Theta)'$. We11 immediately see $\tau(\gamma) = \gamma$ for $\gamma \in {}^{ru}\Delta^+_{\Theta}$. So, we easily have $\sigma_{\gamma} \in {}^{\tau}W$. Hence $W(\Theta) \subseteq {}^{\tau}W$. Since $\tau\Theta = \Theta$, we have $W(\Theta) \subseteq {}^{\tau}W({}^{\tau}\Theta)$. The both $W(\Theta)$ and ${}^{\tau}W({}^{\tau}\Theta)$ are of order eight. Hence $W(\Theta)$ coincides with ${}^{\tau}W({}^{\tau}\Theta)$. So, we have (b) in Corollary 8.2.3. Hence, we can apply Corollary 8.2.3 in this case. Moreover, $W(\Theta)$ can be identified with the little Weyl group for the restricted root system of the real form $\mathfrak{g}_0 = \mathfrak{e}_{6,(-14)}$.

§ 9. Other examples

9.1 A typical example of $W(\Theta)' \subsetneq W(\Theta)$

Let $\mathfrak{g}$ be a complex simple Lie algebra of the type D_{2m} $\quad (n \geqslant 2)$. Then we can choose an orthonormal basis $e_1, ..., e_{2m}$ of $\mathfrak{h}^*$ such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 2m\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 2m\}.$$

If we put $\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i < 2m)$ and $\alpha_{2m} = e_{2m-1} + e_{2m}$, then $\Pi = \{\alpha_1, ..., \alpha_{2m}\}$.

In 9.1, we put $\Theta = \{\alpha_1, ..., \alpha_{2m-2}\}$.

$$\begin{array}{ccccccccc} \odot & - & \odot & - \cdots - & \odot & - & \odot & - & \bigcirc \\ & & & & & & | & & \\ & & & & & & \bigcirc & & \end{array}$$

Then, we have $\mathfrak{a}^*_{\Theta} = \{s(e_1 + \cdots + e_{2m-1}) + te_{2m} \mid s, t \in \mathbb{C}\}$ and $\rho_{\Theta} = \sum_{i=1}^{2m-1}(m - i)e_i$.

In this case, ${}^{ru}\Delta^+_{\Theta} = \emptyset$, $W(\Theta)' = \{e\}$, and $W(\Theta) = \{e, w_{\Theta}w_0\}$ (cf. [15]). We have $w_{\Theta}w_0(\rho_{\Theta} + \mu) = \rho_{\Theta} - \mu$ for all $\mu \in \mathfrak{a}^*_{\Theta}$. We put $\mu_1 = \frac{1}{2}(e_1 + \cdots + e_{2m-1}) + \frac{1}{2}e_{2m}$ and $\mu_2 = -\frac{1}{2}(e_1 + \cdots + e_{2m-1}) + \frac{1}{2}e_{2m}$.

From Jantzen's irreducibility criterion, we have the following lemma.

Lemma 9.1.1. *$M_\Theta(\rho_\Theta + \varepsilon\mu_i)$ is irreducible for each $1 \leqslant i \leqslant 2$ and each $\varepsilon \in \{1, -1\}$.*

Finally, we have the following result.

Proposition 9.1.2. *Let $\nu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \nu$ is regular integral. Then we have $M_\Theta(\rho_\Theta - \nu) \nsubseteq M_\Theta(\rho_\Theta + \nu)$.*

Proof. We assume that $M_\Theta(\rho_\Theta - \nu) \subseteq M_\Theta(\rho_\Theta + \nu)$. We easy to see that there exists some $i \in \{1, 2\}$ and $\varepsilon \in \{1, -1\}$ such that $\langle \varepsilon\mu_i, \gamma\rangle \geqslant 0$ for each $\gamma \in \Sigma_\Theta^+(\nu)$. Applying Proposition 2.4.7, we have $M_\Theta(\rho_\Theta - \varepsilon\mu_i) \subseteq M_\Theta(\rho_\Theta + \varepsilon\mu_i)$. However, it contradicts Lemma 9.1.1. $\square$

9.2 Subregular cases for B_n

Let $\mathfrak{g}$ be a simple Lie algebra of type B_n. We choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ as in 6.3. We also use the notation of the root system in 6.3. We fix $1 \leqslant i \leqslant n-1$ and put $\Theta = \{\alpha_k\}$.

$$\overbrace{\bigcirc - \cdots - \bigcirc}^{k-1} - \odot - \bigcirc - \cdots - \bigcirc \Rightarrow \bigcirc$$

If $2 \leqslant i \leqslant n-2$, then we put $\gamma = e_{k-1} - e_{k+2}$. If $k = n-1$, we put $\gamma = e_{n-2}$.

Then, we have ${}^e\Delta_\Theta = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n, i \neq k, i \neq k+1, j \neq k, j \neq k+1\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant n, i \neq k, i \neq k+1\}$ and ${}^{ru}\Delta_\Theta = {}^e\Delta_\Theta \cup \{\pm(e_i + e_{i+1})\}$. ${}^eW(\Theta)$ is a Weyl group of the type B_{n-2}. We put

$$ {}^eS = \begin{cases} \{s_{\alpha_i} \mid 1 \leqslant i \leqslant k-2\} \cup \{s_{\alpha_i} \mid k+2 \leqslant i \leqslant n\} \cup \{s_\gamma\} & \text{if } k \neq 1 \\ \{s_{\alpha_i} \mid 3 \leqslant i \leqslant n\} & \text{if } k = 1 \end{cases} $$

and denote by $\leqslant_e$ the Bruhat ordering for the Coxeter system $({}^eW(\Theta), {}^eS)$. We can prove the following result in a similar way as Theorem 7.3.5.

Proposition 9.2.1. *Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is regular integral and $\langle \mu, \beta\rangle > 0$ for all $\beta \in {}^e\Sigma_\Theta^+$. Then, for $x, y \in {}^eW(\Theta)$, $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ if and only if $y \leqslant_e x$.*

We remark that $W(\Theta)$ is the direct product of ${}^eW(\Theta)$ and $\{e, s_{e_k+e_{k+1}}\}$. We have the following result.

Proposition 9.2.2. *Let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is regular integral and $\langle \mu, \beta\rangle > 0$ for all $\beta \in {}^{ru}\Sigma_\Theta^+$. Then for $x, y \in {}^eW(\Theta)$, we have $M_\Theta(\rho_\Theta + x\mu) \nsubseteq M_\Theta(\rho_\Theta + ys_{e_k+e_{k+1}}\mu)$ and $M_\Theta(\rho_\Theta + ys_{e_k+e_{k+1}}\mu) \nsubseteq M_\Theta(\rho_\Theta + x\mu)$.*

Sketch of a proof We denote by w_1 the longest element of ${}^eW(\Theta)$. Let V_1 (resp. V_2) be the unique irreducible submodule of $M_\Theta(\rho_\Theta + w_1 s_{e_k+e_{k+1}}\mu)$ (resp. $M_\Theta(\rho_\Theta + w_1\mu)$). From Proposition 1.4.1, V_1 (resp. V_2) is a unique irreducible constituent of $M_\Theta(\rho_\Theta + w_1 s_{e_k+e_{k+1}}\mu)$ (resp. $M_\Theta(\rho_\Theta + w_1\mu)$) of the maximal Gelfand-Kirillov dimension. Applying translation functors successively, we obtain $M_\Theta(-e_{k+1})$ from $M_\Theta(\rho_\Theta + w_1 s_{e_k+e_{k+1}}\mu)$. If we apply the same translation functors, we also obtain $M_\Theta(e_k)$ from $M_\Theta(\rho_\Theta + w_1\mu)$. We can show that $M_\Theta(-e_{k+1})$ and $M_\Theta(e_k)$ are irreducible from Jantzen's irreducibility criterion. So, applying the same translation functors as above, we obtain $M_\Theta(-e_{k+1})$ (resp. $M_\Theta(e_k)$) from V_1 (resp. V_2). This means that $V_1 \not\cong V_2$. From 9.2.1, we have $V_1 \subseteq M_\Theta(\rho_\Theta + y s_{e_k+e_{k+1}}\mu)$ and $V_2 \subseteq M_\Theta(\rho_\Theta + x\mu)$ for any $x, y \in {}^eW(\Theta)$. So, from $V_1 \not\cong V_2$ and Proposition 1.4.1 (3), we have the proposition. $\square$

Corollary 9.2.3. *Let $\mu, \nu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ and $\rho_\Theta + \nu$ are regular integral. We assume that $M_\Theta(\rho_\Theta + \mu) \not\subseteq M_\Theta(\rho_\Theta + \nu)$. Then, it is a composition of elementary homomorphisms.*

Together with Corollary 6.3.7, we have:

Corollary 9.2.4. *We assume that $\mathfrak{g}$ is a complex simple Lie algebra of the type* B_3. *Then, any homomorphism between scalar generalized Verma modules with a regular integral infinitesimal characteris a composition of elementary homomorphisms.*

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